On a conjecture by Pierre Cartier[1]

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Abstract

In [8], Pierre Cartier conjectured that for any non commutative formal power series Φ on $X = \{x_0, x_1\}$ with coefficients in a $\mathbb{Q}$-extension, A, subjected to some suitable conditions, there exists an unique algebra homomorphism φ from the $\mathbb{Q}$-algebra generated by the convergent polyzêtas to A such that Φ is computed from Φ_{KZ} Drinfel'd associator by applying φ to each coefficient. We prove that φ exists and that it is a free Lie exponential over X. Moreover, we give the complete description of the kernel of the polyzêtas and draw some consequences about a structure of the algebra of convergent polyzêtas and about the arithmetical nature of the Euler constant.

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1 Introduction

In 1986, in order to study the linear representations of the braid group B_n [14] coming from the monodromy of the Knizhnik-Zamolodchikov differential equations over $\mathbb{C}_*^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$:

$$dF(z_1, \ldots, z_n) = \Omega_n(z_1, \ldots, z_n)F(z_1, \ldots, z_n), \tag{1}$$

where

$$\Omega_n(z_1, \ldots, z_n) = \frac{1}{2\mathrm{i}\pi} \sum_{1\le i<j\le n} t_{i,j} \frac{d(z_i - z_j)}{z_i - z_j}, \tag{2}$$

and $\{t_{i,j}\}_{i,j\ge 1}$ are noncommutative variables, Drinfel'd introduced a class of formal power series Φ on noncommutative variables over the finite alphabet $X = \{x_0, x_1\}$. Such a power series Φ is called an *associator*.

Since the system (1) is completely integrable, we have [7, 14]

$$d\Omega_n - \Omega_n \wedge \Omega_n = 0. \tag{3}$$

This is equivalent to the fact that the $\{t_{i,j}\}_{i,j\geq 1}$ satisfy the following infinitesimal braid relations

$$
\begin{aligned}
t_{i,j} &= 0 && \text{for} && i = j, && (4)\\
t_{i,j} &= t_{j,i} && \text{for} && i \neq j, && (5)\\
[t_{i,j}, t_{i,k} + t_{j,k}] &= 0 && \text{for distinct} && i, j, k, && (6)\\
[t_{i,j}, t_{k,l}] &= 0 && \text{for distinct} && i, j, k, l. && (7)
\end{aligned}
$$

Example 1. • $\mathcal{T}_2 = \{t_{1,2}\}$.

$$
\begin{aligned}
\Omega_2(z_1, z_2) &= \frac{t_{1,2}}{2\mathrm{i}\pi}\frac{d(z_1 - z_2)}{z_1 - z_2},\\
F(z_1, z_2) &= (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi}.
\end{aligned}
$$

• $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$, $[t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0$.

$$
\begin{aligned}
\Omega_3(z_1, z_2, z_3) &= \frac{1}{2\mathrm{i}\pi}\left[t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right].\\
F(z_1, z_2, z_3) &= G\left(\frac{z_1 - z_2}{z_1 - z_3}\right)(z_1 - z_3)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi},
\end{aligned}
$$

where G satisfies the following fuchsian differential equation with three regular singularities at $0, 1$ and ∞ :

$$(DE) \qquad dG(z) = [x_0\ \omega_0(z) + x_1\ \omega_1(z)]G(z),$$

with

$$
\begin{aligned}
x_0 &:= \frac{t_{1,2}}{2\mathrm{i}\pi} \quad \textit{and} \quad \omega_0(z) := \frac{dz}{z},\\
x_1 &:= -\frac{t_{2,3}}{2\mathrm{i}\pi} \quad \textit{and} \quad \omega_1(z) := \frac{dz}{1 - z}.
\end{aligned}
$$

As already shown by Drinfel'd, the equation (DE) admits, on the simply connected domain $\mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$, two specific solutions

$$G_0(z) \underset{z\rightsquigarrow 0}{\widetilde{\quad}} \exp[x_0 \log(z)] \quad \text{and} \quad G_1(z) \underset{z\rightsquigarrow 1}{\widetilde{\quad}} \exp[-x_1 \log(1 - z)]. \tag{8}$$

Drinfel'd also proved there exists the associator Φ_{KZ} such that

$$G_1^{-1}(z)G_0(z) = \Phi_{KZ}. \tag{9}$$

After that, Lê and Murakami expressed the coefficients of the Drinfel'd associator Φ_{KZ} in terms of *convergent* polyzêtas [22], *i.e.* for $r_1 > 1$,

$$\zeta(r_1, \ldots, r_k) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1^{r_1} \ldots n_k^{r_k}}. \tag{10}$$

In [22], the authors also expressed the *divergent* coefficients as *linear* combinations of convergent polyzêtas via a *regularization process* (see also [38]). This process is one of many ways to regularize the divergent terms.

We will be described the analytical aspects of our regularization process, in Section 4.1, as the *finite part*, of the asymptotic expansions in different scales of comparison [5]. In Section 4.2, it will be seen as the action of the differential Galois group of the polylogarithms (recalled in Section 2.1.2)

$$\mathrm{Li}_{r_1,\ldots,r_k}(z) \quad = \quad \sum_{n_1>\ldots>n_k>0} \frac{z^{n_1}}{n_1^{r_1}\ldots n_k^{r_k}} \tag{11}$$

on the asymptotic expansion of them, at $z = 1$ and in the comparison scale $\{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, and also on the asymptotic expansions, at ∞ and in the comparison scales $\{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, of the harmonic sums (recalled in Section 2.1.1)

$$\mathrm{H}_{r_1,\ldots,r_k}(N) \quad = \quad \sum_{n_1>\ldots>n_k>0}^{N} \frac{1}{n_1^{r_1}\ldots n_k^{r_k}}. \tag{12}$$

This action leads then to the description of the group of associators (Theorem 17) and to a conjecture by Pierre Cartier ([8], conjecture C3). This group is in fact, closely linked to the group of the Chen generating series studied by K.T. Chen to describe the solutions of differential equations [10] and it turns out that each associator regularizes a Chen generating series of the differential forms ω_0 and ω_1 of Example 1 along the integration path on the simply connected domain $\mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$.

In fact, our regularization process is based essentially on two non commutative generating series over the infinite alphabet $Y = \{y_i\}_{i\geq 1}$, which encodes the multi-indices $(r_1, \ldots, r_k)$ by the words $y_{r_1} \ldots y_{r_k}$ over Y^* (the monoid generated by Y), of polylogarithms and of harmonic sums (recalled in Section 2.2.1)

$$\Lambda(z) = \sum_{w\in Y^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) = \sum_{w\in Y^*} \mathrm{H}_w(N)\, w. \tag{13}$$

Through the algebraic combinatorial aspects [45] and the topological aspects [2] of formal series in noncommutative variables, we have already showed the existence of formal series over Y, Z_1 and Z_2 in noncommutative variables with constant terms, such that (see Theorem *à la Abel*, [39])

$$\lim_{z\to 1} \exp\left(y_1 \log\frac{1}{1-z}\right)\Lambda(z) \quad = \quad Z_1, \tag{14}$$

$$\lim_{N\to\infty} \exp\left(\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N) \quad = \quad Z_2. \tag{15}$$

Moreover, Z_1 and Z_2 are equal and stand for the noncommutative generating series of all convergent polyzêtas $\{\zeta(w)\}_{w\in Y^*-y_1Y^*}$ as shown by the factorized form indexed by Lyndon words. This theorem enables, in particular, to explicit the counter-terms eliminating the divergence of the polylogarithms $\{\mathrm{Li}_w(z)\}_{w\in y_1Y^*}$, for $z \to 1$, and of the harmonic sums $\{\mathrm{H}_w(N)\}_{w\in y_1Y^*}$, for

$N \to \infty$, and to calculate the generalized Euler constants associated to the divergent polyzêtas $\{\zeta(w)\}_{w\in y_1Y^*}$ (Corollary 10). This allows also to give, in Section 4.3.2 and via identification of locale coordinations in infinite dimension, the *complete description* of the kernel by its generators, of the following algebra homomorphism[2]

$$\zeta : (\mathbb{Q}\epsilon \oplus (Y - y_1)\mathbb{Q}\langle Y\rangle, ⧢) \longrightarrow (\mathbb{R}, .) \tag{16}$$

$$y_{r_1}\dots y_{r_k} \longmapsto \sum_{n_1>\dots>n_k>0} \frac{1}{n_1^{r_1}\dots n_k^{r_k}}, \tag{17}$$

and the set of *irreducible* polyzêtas (Corollary 16).

Finally, via the *indiscernability* (recalled in Section 2.3) over the group of associators, this study makes precise the structure of the $\mathbb{Q}$-algebra generated by the convergent polyzêtas (Theorem 23) and concludes the challenge of the polynomial relations among polyzêtas indexed by convergent Lyndon words over Y : the generators of this algebra [32, 34, 3, 49].

2 Algebraic structures and analytical studies of harmonic sums and of polylogarithms

2.1 Structures of harmonic sums and of polylogarithms

2.1.1 Quasi-symmetric functions and harmonic sums

Let $\{t_i\}_{i\in\mathbb{N}_+}$ be an infinite set of variables. The elementary symmetric functions η_k and the power sums ψ_k are defined by

$$\eta_k(\underline{t}) = \sum_{n_1>\dots>n_k>0} t_{n_1}\dots t_{n_k} \quad \text{and} \quad \psi_k(\underline{t}) = \sum_{n>0} t_n^k. \tag{18}$$

They are respectively coefficients of the following generating functions

$$\eta(\underline{t} \mid z) = \prod_{i\geq 1}(1 + t_i z) \quad \text{and} \quad \psi(\underline{t} \mid z) = \sum_{i\geq 1} \frac{t_i z}{1 - t_i z}. \tag{19}$$

These generating functions satisfy a Newton identity

$$\frac{d}{dz} \log \eta(\underline{t} \mid z) \quad = \quad \psi(\underline{t} \mid -z). \tag{20}$$

The fundamental theorem from symmetric functions theory asserts that $\{\eta_k\}_{k\geq 0}$ are linearly independent, and provides remarkable identities like (with $\eta_0 = 1$) :

$$\eta_k \quad = \quad \frac{(-1)^k}{k!} \sum_{\substack{s_1,\dots,s_k\geq 0\\ s_1+\dots+ks_k=k+1}} \binom{k}{s_1,\dots,s_k} \left(-\frac{\psi_1}{1}\right)^{s_1} \dots \left(-\frac{\psi_k}{k}\right)^{s_k}. \tag{21}$$

[2]Here, ϵ stands for the empty word over Y.

Let Y be the infinite alphabet $\{y_i\}_{i\geq 1}$ equipped with the order

$$y_1 > y_2 > y_3 > \ldots \tag{22}$$

and let $w = y_{s_1}\ldots y_{s_r} \in Y^*$. The length of w is denoted by $\mid w \mid$.

Let $\mathcal{L}ynY$ be the set of Lyndon words over Y and let $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ and $\{\check{\Sigma}_l\}_{l\in\mathcal{L}ynY}$ be respectively a transcendence basis of the quasi-shuffle algebra $(\mathbb{C}\langle Y\rangle, \uplus)$ and its dual basis, defined by

$$\begin{aligned}
\check{\Sigma}_\epsilon &= 1 && \text{for } l = \epsilon, && (23)\\
\check{\Sigma}_l &= x\check{\Sigma}_u, && \text{for } l = xu \in \mathcal{L}ynY, && (24)\\
\check{\Sigma}_w &= \frac{\check{\Sigma}_{l_1}^{\uplus\, i_1} \uplus \ldots \uplus \check{\Sigma}_{l_k}^{\uplus\, i_k}}{i_1!\ldots i_k!} && \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k. && (25)
\end{aligned}$$

The quasi-symmetric function F_w, of depth $r = \mid w \mid$ and of degree (or weight) $s_1 + \ldots + s_r$, is defined by

$$\mathrm{F}_w(\underline{t}) = \sum_{n_1 > \ldots > n_r > 0} t_{n_1}^{s_1}\ldots t_{n_r}^{s_r}. \tag{26}$$

In particular, $\mathrm{F}_{y_1^k} = \eta_k$ and $\mathrm{F}_{y_k} = \psi_k$. The functions $\{\mathrm{F}_{y_1^k}\}_{k\geq 0}$ are linearly independent and integrating differential equation (20) shows that functions $\mathrm{F}_{y_1^k}$ and F_{y_k} are linked by the formula

$$\sum_{k\geq 0} \mathrm{F}_{y_1^k} z^k = \exp\Bigg(-\sum_{k\geq 1} \mathrm{F}_{y_k}\frac{(-z)^k}{k}\Bigg). \tag{27}$$

Every $\mathrm{H}_w(N)$ can be obtained by specializing, in the quasi-symmetric function F_w, the variables $\{t_i\}_{i\geq 1}$ as follows [41]

$$\forall N \geq i \geq 1, t_i = 1/i \quad \text{and} \quad \forall i > N, t_i = 0. \tag{28}$$

In the same way, for $w \in Y^* - y_1Y^*$, the convergent polyzêta $\zeta(w)$ can be obtained by specializing, in F_w, the variables $\{t_i\}_{i\geq 1}$ as follows [41]

$$\forall N \geq i \geq 1, \qquad t_i = 1/i. \tag{29}$$

The notation F_w is extended by linearity to all polynomials over Y.

If u (resp. v) is a word in Y^*, of length r and of weight[3] p (resp. of length s and of weight q), $\mathrm{F}_{u \uplus v}$ is a quasi-symmetric function of depth $r+s$ and of weight $p+q$, and $\mathrm{F}_{u\uplus v} = \mathrm{F}_u\ \mathrm{F}_v$, where $\uplus$ is the quasi-shuffle product [41]. Hence,

$$\begin{aligned}
&\forall u, v \in Y^*, && \mathrm{H}_{u\uplus v} = \mathrm{H}_u\ \mathrm{H}_v, && (30)\\
\Rightarrow\quad &\forall u, v \in Y^* - y_1Y^*, && \zeta(u \uplus v) = \zeta(u)\ \zeta(v). && (31)
\end{aligned}$$

Remarkable identity (21) can be then seen as

$$y_1^k = \frac{(-1)^k}{k!} \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\ldots+ks_k = k+1}} \binom{k}{s_1,\ldots,s_k} \frac{(-y_1)^{\uplus\, s_1}}{1^{s_1}} \uplus \ldots \uplus \frac{(-y_k)^{\uplus\, s_k}}{k^{s_k}} \tag{32}$$

[3] The weight is as in Equation (26).

2.1.2 Iterated integrals and polylogarithms

Let X be the finite alphabet $\{x_0, x_1\}$ equipped with the order

$$x_0 < x_1 \tag{33}$$

and let

$$\mathcal{C} \quad = \quad \mathbb{C}[z, z^{-1}, (1-z)^{-1}]. \tag{34}$$

The iterated integral over ω_0, ω_1 associated to the word $w = x_{i_1} \cdots x_{i_k}$ over X^* (the monoid generated by X) and along the integration path $z_0 \rightsquigarrow z$ is the following multiple integral defined by

$$\int_{z_0 \rightsquigarrow z} \omega_{i_1} \cdots \omega_{i_k} = \int_{z_0}^{z} \omega_{i_1}(t_1) \int_{z_0}^{t_1} \omega_{i_2}(t_2) \ldots \int_{z_0}^{t_{r-2}} \omega_{i_r}(t_{r-1}) \int_{z_0}^{t_{r-1}} \omega_{i_r}(t_r), \tag{35}$$

where $t_1 \cdots t_{r-1}$ is a subdivision of the path $z_0 \rightsquigarrow z$. In a shortened notation, we denote this integral by $\alpha_{z_0}^{z}(w)$ and[4] $\alpha_{z_0}^{z}(\epsilon) = 1$. One can check that the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}$ is also the value of the iterated integral over ω_0, ω_1 and along the integration path $0 \rightsquigarrow z$ [29, 31] :

$$\mathrm{Li}_w(z) \quad = \quad \alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1). \tag{36}$$

The definition of polylogarithms is extended over the words $w \in X^*$ by putting

$$\mathrm{Li}_{x_0}(z) \quad := \quad \log z. \tag{37}$$

The $\{\mathrm{Li}_w\}_{w \in X^*}$ are $\mathcal{C}$-linearly independent [35, 32]. Thus, the following functions

$$\forall w \in X^*, \quad \mathrm{P}_w(z) \quad := \quad (1-z)^{-1}\,\mathrm{Li}_w(z), \tag{38}$$

are also $\mathbb{C}$-linearly independent. Since, for any $w \in Y^*, \mathrm{P}_w$ is the ordinary generating function of the sequence $\{\mathrm{H}_w(N)\}_{N \geq 0}$ [37] :

$$\mathrm{P}_w(z) \quad = \quad \sum_{N \geq 0} \mathrm{H}_w(N)\; z^N \tag{39}$$

then, as a consequence of the classical isomorphism between convergent Taylor series and their associated sums, the harmonic sums $\{\mathrm{H}_w\}_{w \in Y^*}$ are also $\mathbb{C}$-linearly independent. Firstly, $\ker \mathrm{P} = \{0\}$ and $\ker \mathrm{H} = \{0\}$, and secondly, P is a morphism for the Hadamard product :

$$\mathrm{P}_u(z) \odot \mathrm{P}_v(z) = \sum_{N \geq 0} \mathrm{H}_u(N)\mathrm{H}_v(N) z^N = \sum_{N \geq 0} \mathrm{H}_{u \,\text{⧢}\, v}(N) z^N = \mathrm{P}_{u \,\text{⧢}\, v}(z). \tag{40}$$

[4]Here, ϵ stands for the empty word over X.

Proposition 1 ([37]). *Extended by linearity, the application*

$$\begin{array}{rcl} \mathrm{P} : (\mathbb{C}\langle Y\rangle, \,⧢\,) & \longrightarrow & (\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot)\,, \\ u & \longmapsto & \mathrm{P}_u \end{array}$$

is an isomorphism of algebras. Moreover, the application

$$\begin{array}{rcl} \mathrm{H} : (\mathbb{C}\langle Y\rangle, \,⧢\,) & \longrightarrow & (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, .)\,, \\ u & \longmapsto & \mathrm{H}_u = \{\mathrm{H}_u(N)\}_{N\geq 0} \end{array}$$

is an isomorphism of algebras.

Studying the equivalence between action of $\{(1-z)^l\}_{l\in\mathbb{Z}}$ over $\{\mathrm{P}_w(z)\}_{w\in Y^*}$ and that of $\{N^k\}_{k\in\mathbb{Z}}$ over $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ (see [12]), we have

Theorem 1 ([39]). *The Hadamard $\mathcal{C}$-algebra of $\{\mathrm{P}_w\}_{w\in Y^*}$ can be identified with that of $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$. In the same way, the algebra of harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ with polynomial coefficients can be identified with that of $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$.*

By Identity (32) and by applying the isomorphism H on the set of Lyndon words $\{y_r\}_{1\leq r\leq k}$, we obtain $\mathrm{H}_{y_1^k}$ as polynomials in $\{\mathrm{H}_{y_r}\}_{1\leq r\leq k}$ (which are algebraically independent), and

$$\mathrm{H}_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\ldots+ks_k=k+1}} \frac{(-1)^k}{s_1!\ldots s_k!}\left(-\frac{\mathrm{H}_{y_1}}{1}\right)^{s_1}\ldots\left(-\frac{\mathrm{H}_{y_k}}{k}\right)^{s_k}. \tag{41}$$

2.2 Results *à la Abel* for noncommutative generating series of harmonic sums and of polylogarithms

2.2.1 Generating series of harmonic sums and of polylogarithms

Let $\mathrm{H}(N)$ be the noncommutative generating series of $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ [37] :

$$\mathrm{H}(N) := \sum_{w\in Y^*} \mathrm{H}_w(N)\, w. \tag{42}$$

Theorem 2. *Let*

$$\mathrm{H}_{\mathrm{reg}}(N) := \prod_{l\in\mathcal{L}ynY, l\neq y_1}^{\searrow} e^{\mathrm{H}_{\check{\Sigma}_l}(N)\ \Sigma_l}.$$

Then $\mathrm{H}(N) = e^{\mathrm{H}_{y_1}(N)\ y_1}\mathrm{H}_{\mathrm{reg}}(N)$.

For $l \in \mathcal{L}ynY - \{y_1\}$, the polynomial Σ_l is a finite combination of words in $Y^* - y_1Y^*$. Then we can state the following

Definition 1. *We set* $Z_{⧢} := \mathrm{H}_{\mathrm{reg}}(\infty)$.

The noncommutative generating series of polylogarithms [35, 32]

$$Ł \quad := \quad \sum_{w\in X^*} \mathrm{Li}_w \; w \tag{43}$$

satisfies Drinfel'd's differential equation (DE) of Example 1

$$dŁ \quad = \quad (x_0\omega_0 + x_1\omega_1)Ł \tag{44}$$

with boundary condition [15, 16]

$$Ł(\varepsilon) \quad \widetilde{_{\varepsilon\to 0^+}} \quad e^{x_0\log\varepsilon}. \tag{45}$$

This enables us to prove that Ł is the exponential of a Lie series [35, 32]. Hence,

Proposition 2 (Logarithm of Ł, [38])**.**

$$\begin{aligned}\log Ł(z) \quad &= \quad \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in X^*-\{\epsilon\}} \mathrm{Li}_{u_1\,\sqcup\!\sqcup\,\ldots\,\sqcup\!\sqcup\, u_k}(z)\; u_1\ldots u_k\\ &= \quad \sum_{w\in X^*}\mathrm{Li}_w(z)\;\pi_1(w),\end{aligned}$$

where $\pi_1(w)$ *is the following Lie series*

$$\pi_1(w) \quad = \quad \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in X^*-\{\epsilon\}}\langle w \mid u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, u_k\rangle\; u_1\ldots u_k.$$

Applying a theorem of Ree [44, 45], Ł satisfies Friedrichs criterion [35, 32] :

$$\forall u,v\in X^*,\qquad \mathrm{Li}_{u\sqcup\!\sqcup v} = \mathrm{Li}_u\;\mathrm{Li}_v, \tag{46}$$

$$\Rightarrow\quad \forall u,v\in x_0X^*x_1,\qquad \zeta(u\sqcup\!\sqcup v) = \zeta(u)\;\zeta(v). \tag{47}$$

Proposition 3 (Successive differentiation of Ł, [38])**.** *For any* $l\in\mathbb{N}$*, denoting* $\partial = d/dz$*, we have*

$$\partial^l Ł(z) \quad = \quad P_l(z)Ł(z),$$

where $P_l\in\mathcal{C}\langle X\rangle$ *is defined as follows*

$$P_l(z) \quad = \quad \sum_{\mathrm{wgt}(\mathbf{r})=l}\;\sum_{w\in X^{\deg(\mathbf{r})}}\;\prod_{i=1}^{\deg(\mathbf{r})}\binom{\sum_{j=1}^i r_i + j - 1}{r_i}\tau_{\mathbf{r}}(w),$$

and, for any word $w = x_{i_1}\cdots x_{i_k}$ *and for any composition* $\mathbf{r} = (r_1,\ldots,r_k)$*, of degree* $\deg(\mathbf{r}) = k$ *and of weight* $\mathrm{wgt}(\mathbf{r}) = k + r_1 + \cdots + r_k$*, the polynomial* $\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1})\cdots\tau_{r_k}(x_{i_k})$ *is defined as follows*

$$\forall r\in\mathbb{N},\quad \tau_r(x_0) = \partial^r\frac{x_0}{z} = \frac{-r!x_0}{(-z)^{r+1}} \quad and \quad \tau_r(x_1) = \partial^r\frac{x_1}{1-z} = \frac{r!x_1}{(1-z)^{r+1}}.$$

Let $\{S_l\}_{l\in\mathcal{L}ynX}$ and $\{\check{S}_l\}_{l\in\mathcal{L}ynX}$ be respectively the transcendence basis of the shuffle algebra $(\mathbb{C}\langle X\rangle, ⧢)$ and its dual basis, defined by

$$\check{S}_\epsilon = 1 \quad \text{for } l=\epsilon \tag{48}$$

$$\check{S}_l = x\check{S}_u, \quad \text{for } l = xu \in \mathcal{L}ynX, \tag{49}$$

$$\check{S}_w = \frac{\check{S}_{l_1}^{⧢\, i_1} ⧢ \ldots ⧢ \check{S}_{l_k}^{⧢\, i_k}}{i_1!\ldots i_k!} \quad \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k. \tag{50}$$

Theorem 3 (Factorization of Ł, [35, 32])**.** *Let*

$$Ł_{\mathrm{reg}} := \prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{\mathrm{Li}_{S_l}\,\check{S}_l}.$$

Then $Ł(z) = e^{-x_1\log(1-z)} Ł_{\mathrm{reg}}(z) e^{x_0 \log z}$.

For $l \in \mathcal{L}ynX - \{x_0,x_1\}$, the polynomial S_l is a finite combination of words in $x_0X^*x_1$. Then we can state the following

Definition 2 ([35, 32])**.** *We set* $Z_{⧢} := Ł_{\mathrm{reg}}(1)$.

In the definitions 1 and 2 only convergent polyzêtas arise and these noncommutative generating series will induce, in Section 4.1, two algebra morphisms of regularization as shown in the theorems 12 and 13 respectively. Hence, these power series are quite different of those given in [22] or in [43] (the last is based on [4], see [8]) needing a regularization process.

2.2.2 Asymptotic expansions by noncommutative generating series and regularized Chen's generating series

Let $\rho_{1-z}, \rho_{1-\frac{1}{z}}$ and $\rho_{\frac{1}{z}}$ [36, 32] be three monoid morphisms verifying

$$\rho_{1-z}(x_0) = -x_1 \quad \text{and} \quad \rho_{1-z}(x_1) = -x_0, \tag{51}$$

$$\rho_{1-1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1-1/z}(x_1) = -x_0 \tag{52}$$

$$\rho_{1/z}(x_0) = -x_0 + x_1 \quad \text{and} \quad \rho_{1/z}(x_0) = x_1. \tag{53}$$

One also has [36, 32]

$$Ł(1-z) = e^{x_0\log(1-z)}\rho_{1-z}[Ł_{\mathrm{reg}}(z)]e^{-x_1\log z} Z_{⧢}, \tag{54}$$

$$Ł(1-1/z) = e^{x_0\log(1-z)}\rho_{1-\frac{1}{z}}[Ł_{\mathrm{reg}}(z)]e^{-x_1\log z}\rho_{1-1/z}(Z_{⧢}^{-1})e^{\mathrm{i}\pi\, x_0} \tag{55}$$

$$Ł(1/z) = e^{-x_1\log(1-z)}\rho_{1/z}[Ł_{\mathrm{reg}}(z)]e^{(-x_0+x_1)\log z}\rho_{1/z}(Z_{⧢}^{-1})e^{\mathrm{i}\pi\, x_1} Z_{⧢} \tag{56}$$

Thus, (45) and (54) yield [36, 32]

$$Ł(z) \underset{z\to 0}{\widetilde{\ \ }} \exp(x_0\log z) \quad \text{and} \quad Ł(z) \underset{z\to 1}{\widetilde{\ \ }} \exp(-x_1\log(1-z))\, Z_{⧢}. \tag{57}$$

Let us call $\mathrm{LI}_{\mathcal{C}}$ the smallest algebra containing $\mathcal{C}$, closed under derivation and under integration with respect to ω_0 and ω_1. It is the $\mathcal{C}$-module generated by the polylogarithms $\{\mathrm{Li}_w\}_{w\in X^*}$.

Let $\Pi_Y : \mathrm{LI}_{\mathcal{C}}\langle\langle X\rangle\rangle \longrightarrow \mathrm{LI}_{\mathcal{C}}\langle\langle Y\rangle\rangle$ be a projector such that for any $f \in \mathrm{LI}_{\mathcal{C}}$ and $w \in X^*$, $\Pi_Y(f\ wx_0) = 0$. Then [39]

$$\Lambda(z) = \Pi_Y \mathrm{Ł}(z) \widetilde{_{z\to 1}} \exp\left(y_1 \log \frac{1}{1-z}\right)\Pi_Y Z_{\sqcup\!\sqcup}. \tag{58}$$

Since the coefficient of z^N in the ordinary Taylor expansion of $\mathrm{P}_{y_1^k}$ is $\mathrm{H}_{y_1^k}(N)$ then let [39]

$$\mathrm{Mono}(z) := e^{-(x_1+1)\log(1-z)} = \sum_{k\geq 0} \mathrm{P}_{y_1^k}(z)\ y_1^k \tag{59}$$

$$\mathrm{Const} := \sum_{k\geq 0} \mathrm{H}_{y_1^k}\ y_1^k = \exp\left(-\sum_{k\geq 1} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right). \tag{60}$$

Proposition 4 ([39])**.**

$$\Pi_Y \mathrm{P}(z) \widetilde{_{z\to 1}} \mathrm{Mono}(z)\Pi_Y Z_{\sqcup\!\sqcup} \quad \textit{and} \quad \mathrm{H}(N) \widetilde{_{N\to\infty}} \mathrm{Const}(N)\Pi_Y Z_{\sqcup\!\sqcup}.$$

Proof. Let μ be the morphism verifying $\mu(x_0) = x_1$ and $\mu(x_1) = x_0$. Then, by Theorem 3, the noncommutative generating series of $\{\mathrm{P}_w\}_{w\in X^*}$ is given by

$$\begin{aligned}
\mathrm{P}(z) &= (1-z)^{-1}\mathrm{Ł}(z)\\
&= e^{-(x_1+1)\log(1-z)}\mathrm{Ł}_{\mathrm{reg}}(z)e^{x_0\log z}\\
&= e^{x_0\log z}\mu[\mathrm{Ł}_{\mathrm{reg}}(1-z)]e^{-(x_1+1)\log(1-z)}Z_{\sqcup\!\sqcup}\\
&= e^{x_0\log z}\mu[\mathrm{Ł}_{\mathrm{reg}}(1-z)]\mathrm{Mono}(z)Z_{\sqcup\!\sqcup}.
\end{aligned}$$

Thus, $\mathrm{P}(z) \widetilde{_{z\to 0}} e^{x_0\log z}$and $\mathrm{P}(z) \widetilde{_{z\to 1}} \mathrm{Mono}(z)Z_{\sqcup\!\sqcup}$ yielding the expected results. $\square$

As consequence of (58)-(60) and of Proposition 4, one gets

Theorem 4 (*à la Abel*, [39])**.**

$$\lim_{z\to 1} \exp\left(y_1 \log\frac{1}{1-z}\right)\Lambda(z) = \lim_{N\to\infty} \exp\left(\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N) = \Pi_Y Z_{\sqcup\!\sqcup}.$$

Therefore, the knowledge of the ordinary Taylor expansion at 0 of the polylogarithmic functions $\{\mathrm{P}_w(1-z)\}_{w\in X^*}$ gives

Theorem 5 ([12])**.** *For all $g \in \mathcal{C}\{\mathrm{P}_w\}_{w\in Y^*}$, there exists algorithmically computable $c_j \in \mathbb{C}, \alpha_j \in \mathbb{Z}, \beta_j \in \mathbb{N}$ and $b_i \in \mathbb{C}, \eta_i \in \mathbb{Z}, \kappa_i \in \mathbb{N}$ such that*

$$\begin{aligned}
g(z) &\sim \sum_{j=0}^{+\infty} c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z) \quad \textit{for} \quad z \to 1,\\
[z^n]g(z) &\sim \sum_{i=0}^{+\infty} b_i n^{\eta_i}\log^{\kappa_i}(n) \qquad\qquad \textit{for} \quad n \to \infty.
\end{aligned}$$

Definition 3. *Let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by convergent polyzêtas and let $\mathcal{Z}'$ be the*[5] *$\mathbb{Q}[\gamma]$-algebra generated by $\mathcal{Z}$.*

Corollary 1 ([12])**.** *There exists algorithmically computable $c_j \in \mathcal{Z}, \alpha_j \in \mathbb{Z}, \beta_j \in \mathbb{N}$ and $b_i \in \mathcal{Z}', \kappa_i \in \mathbb{N}, \eta_i \in \mathbb{Z}$ such that*

$$\begin{aligned}
\forall w \in Y^*, \quad \mathrm{P}_w(z) \quad &\sim \quad \sum_{j=0}^{+\infty} c_j (1-z)^{\alpha_j} \log^{\beta_j}(1-z) \quad \textit{for} \quad z \to 1, \\
\forall w \in Y^*, \quad \mathrm{H}_w(N) \quad &\sim \quad \sum_{i=0}^{+\infty} b_i N^{\eta_i} \log^{\kappa_i}(N) \qquad\qquad \textit{for} \quad N \to +\infty.
\end{aligned}$$

The Chen generating series along the path $z_0 \rightsquigarrow z$, associated to ω_0, ω_1 is the following power series

$$S_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \langle S \mid w \rangle \, w \quad \text{with} \quad \langle S \mid w \rangle = \alpha_{z_0}^{z}(w) \tag{61}$$

which solves the differential equation (44) with the initial condition

$$S_{z_0 \rightsquigarrow z_0} \quad = \quad 1. \tag{62}$$

Thus, $S_{z_0 \rightsquigarrow z}$ and $\mathrm{Ł}(z)\mathrm{Ł}(z_0)^{-1}$ satisfy the same differential equation taking the same value at z_0 and

$$S_{z_0 \rightsquigarrow z} \quad = \quad \mathrm{Ł}(z)\mathrm{Ł}(z_0)^{-1}. \tag{63}$$

Any Chen generating series $S_{z_0 \rightsquigarrow z}$ is group like [44] and depends only on the homotopy class of $z_0 \rightsquigarrow z$ [10]. The product of two Chen generating series $S_{z_1 \rightsquigarrow z_2}$ and $S_{z_0 \rightsquigarrow z_1}$ is the Chen generating series

$$S_{z_0 \rightsquigarrow z_2} \quad = \quad S_{z_1 \rightsquigarrow z_2} S_{z_0 \rightsquigarrow z_1}. \tag{64}$$

Let $\varepsilon \in]0,1[$ and let $z_i = \varepsilon \exp(\mathrm{i}\theta_i)$, for $i = 0, 1$. We set $\theta = \theta_1 - \theta_0$. Let $\Gamma_0(\varepsilon, \theta)$ (resp. $\Gamma_1(\varepsilon, \theta)$) be the path turning around 0 (resp. 1) in the positive direction from z_0 to z_1. By induction on the length of w, one has

$$|\langle S_{\Gamma_i(\varepsilon,\theta)} \mid w \rangle| \quad = \quad (2\varepsilon)^{|w|_{x_i}} \theta^{|w|} / |w|!, \tag{65}$$

where $|w|_{x_i}$, denotes the number of occurrences of letter x_i in w, for $i = 0, 1$. For $\varepsilon \to 0^+$, these estimations yield

$$S_{\Gamma_i(\varepsilon,\theta)} \quad = \quad e^{\mathrm{i}\theta x_i} + o(\varepsilon). \tag{66}$$

[5] Here, γ stands for the Euler constant

$$\gamma = .57721566490153286060651209008240243104215933593992359880576723488486 77\ldots$$

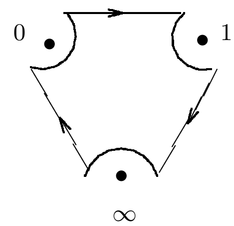

Figure 1: Hexagonal path

In particular, if $\Gamma_0(\varepsilon)$ (resp. $\Gamma_1(\varepsilon)$) is a circular path of radius ε turning around 0 (resp. 1) in the positive direction, starting at $z = \varepsilon$ (resp. $1-\varepsilon$), then, by the noncommutative residu theorem [35, 32], we get

$$S_{\Gamma_0(\varepsilon)} = e^{2\mathrm{i}\pi x_0} + o(\varepsilon) \quad \text{and} \quad S_{\Gamma_1(\varepsilon)} = e^{-2\mathrm{i}\pi x_1} + o(\varepsilon). \tag{67}$$

Finally, the asymptotic behaviors of Ł on (57) give

Proposition 5 ([32, 35])**.** *We have* $S_{\varepsilon\rightsquigarrow 1-\varepsilon} \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{-x_1 \log\varepsilon} Z_{\sqcup\!\sqcup}\ e^{-x_0\log\varepsilon}$.

In other terms, $Z_{\sqcup\!\sqcup}$ is the regularized Chen generating series $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ of differential forms ω_0 and ω_1 : $Z_{\sqcup\!\sqcup}$ is the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $e^{x_1\log\varepsilon}\ S_{\varepsilon\rightsquigarrow 1-\varepsilon}\ e^{x_0\log\varepsilon}$: the concatenation of $e^{x_0\log\varepsilon}$ and then $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1\log\varepsilon}$.

Proposition 6. *We have*

$$\prod^{\searrow}_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}} e^{\zeta(\check{l})l} \quad = \quad e^{\mathrm{i}\pi x_0} \prod^{\searrow}_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}} e^{\zeta(\check{l})\rho_{1-1/z}(l)} e^{\mathrm{i}\pi(-x_0+x_1)} \prod^{\searrow}_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}} e^{\zeta(\check{l})\rho^2_{1-1/z}(l)} e^{-\mathrm{i}\pi x_1},$$

where the morphism $\rho_{1-1/z}$ is given in Section 2.2.2.

Proof. Following the hexagonal path given in Figure 1, one has [36, 32]

$$(S_{\varepsilon\rightsquigarrow 1-\varepsilon} e^{\mathrm{i}\pi x_0})\rho_{1-1/z}(S_{\varepsilon\rightsquigarrow 1-\varepsilon} e^{\mathrm{i}\pi x_0})\rho^2_{1-1/z}(S_{\varepsilon\rightsquigarrow 1-\varepsilon} e^{\mathrm{i}\pi x_0}) = 1 + O(\sqrt{\varepsilon}).$$

By Proposition 5, it follows the "hexagonal relation" (see [15, 16, 36, 32]) which is a relateur in $dm(A)$:

$$\begin{aligned} & Z_{\sqcup\!\sqcup} e^{\mathrm{i}\pi x_0}\rho_{1-1/z}(Z_{\sqcup\!\sqcup}) e^{\mathrm{i}\pi(-x_0+x_1)}\rho^2_{1-1/z}(Z_{\sqcup\!\sqcup}) e^{-\mathrm{i}\pi x_1} = 1, \\ \iff \quad & e^{\mathrm{i}\pi x_0}\rho_{1-1/z}(Z_{\sqcup\!\sqcup}) e^{\mathrm{i}\pi(-x_0+x_1)}\rho^2_{1-1/z}(Z_{\sqcup\!\sqcup}) e^{-\mathrm{i}\pi x_1} = Z^{-1}_{\sqcup\!\sqcup}. \end{aligned}$$

It follows then the expected result. $\square$

2.3 Indiscernability over a class of formal power series

2.3.1 Residual calculus and representative series

Definition 4. *Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$ and let $P \in \mathbb{Q}\langle X\rangle$.*

The left residual *(resp.* right residual*) of S by P, is the formal power series $P \triangleleft S$ (resp. $S \triangleright P$) in $\mathbb{Q}\langle\langle X\rangle\rangle$ defined by :*

$$\langle P \triangleleft S \mid w\rangle = \langle S \mid wP\rangle \quad (resp. \quad \langle S \triangleright P \mid w\rangle = \langle S \mid Pw\rangle).$$

We straightforwardly get, for any $P, Q \in \mathbb{Q}\langle X\rangle$:

$$P \triangleleft (Q \triangleleft S) = PQ \triangleleft S,\ (S \triangleright P) \triangleright Q = S \triangleright PQ,\ (P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q). \quad (68)$$

In case $x, y \in X$ and $w \in X^*$, we get[6] :

$$x \triangleleft (wy) = \delta_x^y w \quad \text{and} \quad xw \triangleright y = \delta_x^y w. \quad (69)$$

Lemma 1 (Reconstruction lemma)**.** *Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$. Then*

$$S = \langle S \mid \epsilon\rangle + \sum_{x\in X} x(S \triangleright x) = \langle S \mid \epsilon\rangle + \sum_{x\in X} (x \triangleleft S)x.$$

Lemma 2. *The left (resp. right) residual* by a letter x *is a derivation of* $(\mathbb{Q}\langle\langle X\rangle\rangle, ⧢)$ *:*

$$x \triangleleft (u ⧢ v) = (x \triangleleft u) ⧢ v + u ⧢ (x \triangleleft v), \quad (u ⧢ v) \triangleright x = (u \triangleright x) ⧢ v + u ⧢ (v \triangleright x).$$

Proof. Use the recursive definitions of the shuffle product. □

Consequently,

Lemma 3. *For any Lie polynomial $Q \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$, the linear maps "$Q\triangleleft$" and "$\triangleright Q$" are derivations on* $(\mathbb{Q}[\mathcal{L}ynX], ⧢)$*.*

Proof. For any $l, l_1, l_2 \in \mathcal{L}ynX$, we have

$$\hat{l} \triangleleft (l_1 ⧢ l_2) = l_1 ⧢ (\hat{l} \triangleleft l_2) + (\hat{l} \triangleleft l_1) ⧢ l_2 = l_1\delta_{l_2}^{\hat{l}} + \delta_{l_1}^{\hat{l}} l_2,$$
$$(l_1 ⧢ l_2) \triangleright \hat{l} = l_1 ⧢ (l_2 \triangleright \hat{l}) + (l_1 \triangleright \hat{l}) ⧢ l_2 = l_1\delta_{l_2}^{\hat{l}} + \delta_{l_1}^{\hat{l}} l_2.$$

□

Lemma 4. *For any Lyndon word $l \in \mathcal{L}yn - \{x_0, x_1\}$, one has*

$$x_1 \triangleleft l = l \triangleright x_0 = 0 \quad \textit{and} \quad x_1 \triangleleft \check{S}_l = \check{S}_l \triangleright x_0 = 0.$$

Proof. Since $x_1\triangleleft$ and $\triangleright x_0$ are derivations and for any $l \in \mathcal{L}ynX - \{x_0, x_1\}$, the polynomial $\check{S}_l$ belongs to $x_0\mathbb{Q}\langle X\rangle x_1$ then it follows the expected results. □

[6] For any words u and $v \in X^*$, if $u = v$ then $\delta_u^v = 1$ else 0.

Theorem 6 (On representative series)**.** *The following properties are equivalent for any series* $S \in \mathbb{Q}\langle\langle X \rangle\rangle$ *:*

1. *The left* $\mathbb{C}$*-module* $Res_g(S) = span\{w \triangleleft S \mid w \in X^*\}$ *is finite dimensional.*
2. *The right* $\mathbb{C}$*-module* $Res_d(S) = span\{S \triangleright w \mid w \in X^*\}$ *is finite dimensional.*
3. *There are matrices* $\lambda \in \mathcal{M}_{1,n}(\mathbb{Q})$*,* $\eta \in \mathcal{M}_{n,1}(\mathbb{Q})$ *and a representation of* X^* *in* $\mathcal{M}_{n,n}$*, such that*

$$S = \sum_{w \in X^*} [\lambda\mu(w)\eta]\, w = \lambda\Big(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mu(S_l)\, \check{S}_l}\Big)\eta.$$

A series that safisfies the items of this theorem will be called *representative series*. This concept can be found in [1, 42, 13]. The two first items are in [18, 24]. The third item can be deduced from [9, 11] for example and it was used to factorize first time, by Lyndon words, the output of bilinear and analytical dynamical systems respectively in [26, 27] and to study polylogarithms, hypergeometric functions and associated functions in [29, 31, 38]. The dimension of $Res_g(S)$ is equal to that of $Res_d(S)$, and to the minimal dimension of a representation satisfying the third point of Theorem 6. This rank is then equal to the rank of the Hankel matrix of S, that is the infinite matrix $(\langle S \mid uv\rangle)_{u,v\in X}$ indexed by $X^* \times X^*$ and is also called *Hankel rank* of S [18, 24] :

Definition 5 ([18, 24])**.** *The* Hankel rank *of a formal power series* $S \in \mathbb{C}\langle\langle X \rangle\rangle$ *is the dimension of the vector space*

$$\{S \triangleright \Pi \mid \Pi \in \mathbb{C}\langle X \rangle\}, \quad (resp. \quad \{\Pi \triangleleft S \mid \Pi \in \mathbb{C}\langle X \rangle\}.$$

The triplet (λ, μ, η) is called a *linear representation* of S. We define the minimal representation[7] of S as being a representation of S of minimal dimension.

For any proper series S, the following power series is called "star of S"

$$S^* = 1 + S + S^2 + \ldots + S^n + \ldots. \tag{70}$$

Definition 6 ([2, 47])**.** *A series S is called* rational *if it belongs to the closure in* $\mathbb{Q}\langle\langle X \rangle\rangle$ *of the noncommutative polynomial algebra by sum, product, and star operation of* proper[8] *elements. The set of rational power series will be denoted by* $\mathbb{Q}^{\mathrm{rat}}\langle\langle X \rangle\rangle$.

Lemma 5. *For any noncommutative rational series (resp. polynomial) R and for any polynomial P, the left and right residuals of R by P are also rational (resp. polynomial).*

Theorem 7 (Schützenberger, [2, 47])**.** *Any noncommutative power series is representative if and only if it is rational.*

[7] It can be shown that all minimal representations are isomorphics (see [2]).

[8] A series S is said to be proper if $\langle S \mid \epsilon\rangle = 0$.

2.3.2 Continuity and indiscernability

Definition 7 ([25, 39]). *Let $\mathcal{H}$ be a class of formal power series over X. Let $S \in \mathbb{C}\langle\langle X\rangle\rangle$.*

1. *S is said to be* continuous[9] *over $\mathcal{H}$ if for any $\Phi \in \mathcal{H}$, the following sum, denoted $\langle S \parallel \Phi\rangle$, is convergent in norm*
$$\sum_{w\in X^*} \langle S \mid w\rangle\langle \Phi \mid w\rangle.$$
The set of continuous power series over $\mathcal{H}$ will be denoted by $\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$.

2. *S is said to be* indiscernable[10] *over $\mathcal{H}$ if and only if*
$$\forall \Phi \in \mathcal{H}, \qquad \langle S \parallel \Phi\rangle = 0.$$

Let ρ be the monoid morphism verifying $\rho(x_0) = x_1$ and $\rho(x_1) = x_0$ and let $\hat{w} = \rho(\tilde{w})$, where $\tilde{w}$ is the mirror of w.

Lemma 6. *Let $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$. If $\langle S \parallel Z_{\sqcup\!\sqcup}\rangle = 0$ then $\langle \hat{S} \parallel Z_{\sqcup\!\sqcup}\rangle = 0$, where*
$$\hat{S} := \sum_{w\in X^*} \langle S \mid w\rangle\ \hat{w}.$$

Proof. For any $w \in x_0X^*x_1$, by "duality relation", one has (see [40, 50, 36])
$$\zeta(\hat{w}) = \zeta(w), \quad \text{or equivalently} \quad Z_{\sqcup\!\sqcup} = \hat{Z}_{\sqcup\!\sqcup} := \sum_{w\in X^*} \langle Z_{\sqcup\!\sqcup} \mid w\rangle\ \hat{w}.$$
Use the fact
$$\langle \hat{S} \parallel Z_{\sqcup\!\sqcup}\rangle = \sum_{\hat{w}\in X^*} \langle S \mid \hat{w}\rangle\langle Z_{\sqcup\!\sqcup} \mid \hat{w}\rangle = \sum_{w\in X^*} \langle S \mid w\rangle\langle Z_{\sqcup\!\sqcup} \mid w\rangle,$$
one gets finally the expected result. □

Lemma 7. *Let $\mathcal{H}$ be a monoid containing $\{e^{t\,x}\}_{x\in X}^{t\in\mathbb{R}}$. Let $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ being indiscernable over $\mathcal{H}$. Then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ and they are indiscernable over $\mathcal{H}$.*

Proof. Let us calculate $\langle x \triangleleft S \parallel \Phi\rangle = \langle S \parallel \Phi x\rangle$ and $\langle S \triangleright x \parallel \Phi\rangle = \langle S \parallel x\Phi\rangle$. Since
$$\lim_{t\to 0} \frac{e^{t\,x} - 1}{t} = x \quad \text{and} \quad \lim_{t\to 0} \frac{e^{t\,x} - 1}{t} = x$$
then, for any $\Phi \in \mathcal{H}$, by uniform convergence, one has
$$\langle S \parallel \Phi x\rangle = \langle S \parallel \lim_{t\to 0} \Phi\frac{e^{t\,x} - 1}{t}\rangle = \lim_{t\to 0}\langle S \parallel \Phi\frac{e^{t\,x} - 1}{t}\rangle,$$
$$\langle S \parallel x\Phi\rangle = \langle S \parallel \lim_{t\to 0} \frac{e^{t\,x} - 1}{t}\Phi\rangle = \lim_{t\to 0}\langle S \parallel \frac{e^{t\,x} - 1}{t}\Phi\rangle.$$

[9] See [25, 39], for a convergence criterion and an example of continuous generating series.
[10] Here, we adapt this notion developped in [25] via the residual calculus.

Since S is indiscernable over $\mathcal{H}$ then

$$\langle S \parallel \Phi x\rangle = \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi e^{t\,x}\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0,$$
$$\langle S \parallel x\Phi\rangle = \lim_{t\to 0}\frac{1}{t}\langle S \parallel e^{t\,x}\Phi\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0.$$

□

Proposition 7. *Let $\mathcal{H}$ be a monoid containing $\{e^{t\,x}\}_{x\in X}^{t\in\mathbb{R}}$. The power series $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ is indiscernable over $\mathcal{H}$ if and only if $S = 0$.*

Proof. If $S = 0$ then it is immediate that S is indiscernable over $\mathcal{H}$. Conversely, if S is indiscernable over $\mathcal{H}$ then by Lemma 7, for any word $w \in X^*$, by induction on the length of w, $w \triangleleft S$ is indiscernable over $\mathcal{H}$ and then in particular,

$$\langle w \triangleleft S \parallel \mathrm{Id}_{\mathcal{H}}\rangle = \langle S \mid w\rangle = 0.$$

In other words, $S = 0$. □

3 Polysystem

3.1 Polysystem and convergence criterium

3.1.1 Serial estimates from above

In all the sequel, we follow the notations of [2, 45].

Here, generalizing a little, $\mathbb{K}$ is supposed a $\mathbb{C}$-algebra and a complete normed vector space equipped with a norm denoted by $\|.\|$.

For any $n \in \mathbb{N}, X^{\geq n}$ denotes the set of words over X of length greater than or equal to n. The set of formal power series (resp. polynomials) on X, is denoted by $\mathbb{K}\langle\langle X\rangle\rangle$ (resp. $\mathbb{K}\langle X\rangle$).

Definition 8 ([25, 39])**.** *Let ξ, χ be real positive functions over X^*.*

Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$.

1. *S will be said $\xi-$*exponentially bounded from above *if it verifies*[11]

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\xi(w)/|w|!.$$

We denote by $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ which are $\xi-$exponentially bounded from above.

2. *S verifies the $\chi-$*growth condition *if it satisfies*

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\chi(w)\,|w|!.$$

We denote by $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ verifying the $\chi-$growth condition.

[11]Here, $|w|$ denotes the length of $w \in X^*$.

Lemma 8. *We have*

$$R=\sum_{w\in X^*}|\,w\,|!\,w \quad\Rightarrow\quad \langle R^{\sqcup\!\sqcup 2}\mid w\rangle=\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u\,\sqcup\!\sqcup\,v)\ni w}}|\,u\,|!\,|\,v\,|!\le 2^{|w|}\,|\,w\,|!.$$

Proof. One has

$$\begin{aligned}\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u\,\sqcup\!\sqcup\,v)\ni w}}|\,u\,|!\,|\,v\,|! &= \sum_{k=0}^{|w|}\sum_{\substack{|u|=k,|v|=|w|-k\\ \mathrm{supp}(u\,\sqcup\!\sqcup\,v)\ni w}}k!(|\,w\,|-k)!\\ &= \sum_{k=0}^{|w|}\binom{|\,w\,|}{k}k!(|\,w\,|-k)!\\ &= \sum_{k=0}^{|w|}|\,w\,|!\\ &= (1+|\,w\,|)\,|\,w\,|!.\end{aligned}$$

Since, by induction on the length of w, one has $1+|\,w\,|\le 2^{|w|}$ then we get the expected result. □

Proposition 8. *Let S_1 and S_2 verifying the growth condition. Then S_1+S_2 and $S_1 \sqcup\!\sqcup S_2$ also verifies the growth condition.*

Proof. The proof for S_1+S_2 is immediate.

Next, since $\|\langle S_i\mid w\rangle\|\le K_i\chi_i(w)\,|\,w\,|!$, for $i=1$ or 2 and for $w\in X^*$, then[12]

$$\begin{aligned}\langle S_1\sqcup\!\sqcup S_2\mid w\rangle &= \sum_{\mathrm{supp}(u\,\sqcup\!\sqcup\,v)\ni w}\langle S_1\mid u\rangle\langle S_2\mid v\rangle,\\ \Rightarrow\quad \|\langle S_1\sqcup\!\sqcup S_2\mid w\rangle\| &\le K_1K_2\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u\,\sqcup\!\sqcup\,v)\ni w}}(\chi_1(u)\,|\,u\,|!)(\chi_2(v)\,|\,v\,|!).\end{aligned}$$

Let $K=K_1K_2$ and let χ be a real positive function over X^* such that

$$\forall w\in X^*,\qquad \chi(w)=\max\{\chi_1(u)\chi_2(v)\mid u,v\in X^* \text{ and } \mathrm{supp}(u\sqcup\!\sqcup v)\ni w\}.$$

With the notations in Lemma 8, we get

$$\|\langle S_1\sqcup\!\sqcup S_2\mid w\rangle\| \le K\chi(w)\langle R^{\sqcup\!\sqcup 2}\mid w\rangle.$$

Hence, $S_1\sqcup\!\sqcup S_2$ verifies the χ'-growth condition with χ' defined as, for $w\in X^*$, $\chi'(w)=2^{|w|}\chi(w)$. □

Definition 9. *Let ξ be a real positive function defined over X^*, S will be said* ξ-exponentially continuous *if it is continuous over $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$.*

We denote by $\mathbb{K}^{\xi-ec}\langle\langle X\rangle\rangle$ the set of formal power series which are ξ-exponentially continuous.

[12] $\langle S_1\sqcup\!\sqcup S_2\mid w\rangle$ is the coefficient of the word w in the power series $S_1\sqcup\!\sqcup S_2$.

Lemma 9 ([25, 39])**.** *For any real positive function ξ defined over X^*, we have $\mathbb{K}\langle X\rangle \subset \mathbb{K}^{\xi-ec}\langle\langle X\rangle\rangle$. Otherwise, for $\xi = 0$, we get $\mathbb{K}\langle X\rangle = \mathbb{K}^{0-ec}\langle\langle X\rangle\rangle$. Hence, any polynomial is $0-$exponentially continuous.*

Proposition 9 ([25, 39])**.** *Let ξ, χ be a real positive functions over X^*. Let $P \in \mathbb{K}\langle X\rangle$.*

1. *Let $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$. The right residual of S by P belongs to $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$.*
2. *Let $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$. The concatenation SR belongs to $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$.*

Proof. 1. Since $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\xi(w)/|w|!.$$

If $u \in \mathrm{supp}(P) := \{w \in X^* \mid \langle P \mid w\rangle \neq 0\}$ then, for any $w \in X^*$, one has $\langle S \triangleright u \mid w\rangle = \langle S \mid uw\rangle$ and $S \triangleright u$ belongs to $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$:

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \triangleright u \mid w\rangle\| \leq [K\xi(u)]\xi(w)/|w|!.$$

It follows then $S \triangleright P$ is $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ by taking $K_1 = K \max_{u\in\mathrm{supp}(P)} \xi(u)$.

2. Since $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\chi(w)\,|w|!.$$

Let $v \in \mathrm{supp}(P)$ such that $v \neq \epsilon$. Since, for any $w \in X^*$, one has $\langle Rv \mid w\rangle = \langle R \mid v \triangleleft w\rangle$ and Rv belongs to $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$:

$$\begin{aligned}\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|\langle R \mid v \triangleleft w\rangle\| &\leq K\chi(v \triangleleft w)(|w| - |v|)! \\ &\leq K\,|w|\,\chi(w)/\chi(v).\end{aligned}$$

Note that if $v \triangleleft w = 0$ then $\langle Rv \mid w\rangle = 0$ and the previous conclusion holds. It follows then RP is $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ by taking $K_2 = K \min_{v\in\mathrm{supp}(P)} \chi(v)^{-1}$. □

Proposition 10 ([25, 39])**.** *Two real positive morphisms over X^*, ξ and χ are assumed to verify the condition*

$$\sum_{x\in X} \chi(x)\xi(x) \quad < \quad 1.$$

Then for any $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$, F is continuous over $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$.

Proof. If ξ, χ verify the upper bound condition then the following power series is well defined

$$\sum_{w\in X^*} \chi(w)\xi(w) \quad = \quad \Big(\sum_{x\in X} \chi(x)\xi(x)\Big)^*.$$

If $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ and $C \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ then there exists $K_i \in \mathbb{R}_+$ and $n_i \in \mathbb{N}$ such that for any $w \in X^{\geq n_i}, i = 1, 2$, one has

$$\|\langle F \mid w\rangle\| \leq K_1\chi(w)\,|w|! \quad \text{and} \quad \|\langle C \mid w\rangle\| \leq K_2\xi(w)/|w|!.$$

Hence,

$$\forall w \in X^*, |w| \geq \max\{n_1, n_2\}, \qquad \|\langle F|w\rangle\langle C|w\rangle\| \leq K_1K_2\chi(w)\xi(w).$$

Therefore,

$$\sum_{w\in X^*} \|\langle F|w\rangle\langle C|w\rangle\| \leq K_1K_2 \sum_{w\in X^*} \chi(w)\xi(w) = K_1K_2\Big(\sum_{x\in X}\chi(x)\xi(x)\Big)^*.$$

□

3.1.2 Upper bounds *à la* Cauchy

Let $q_1, \ldots, q_n$ be commutative indeterminates over $\mathbb{C}$. The algebra of formal power series (resp. polynomials) over $\{q_1, \ldots, q_n\}$ with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[\![q_1, \ldots, q_n]\!]$ (resp. $\mathbb{C}[q_1, \ldots, q_n]$).

Definition 10. *Let*

$$f \quad = \sum_{i_1,\ldots,i_n\geq 0} f_{i_1,\ldots,i_n} q_1^{i_1}\ldots q_n^{i_n} \quad \in \quad \mathbb{C}[\![q_1, \ldots, q_n]\!].$$

We set

$$\begin{array}{rcl} E(f) & = & \{\rho \in \mathbb{R}_+^n : \exists C_f \in \mathbb{R}_+ \textit{ s.t. } \forall i_1, \ldots, i_n \geq 0, |f_{i_1,\ldots,i_n}|\,\rho_1^{i_1}\ldots\rho_n^{i_n} \leq C_f\} \\ \breve{E}(f) & : & \textit{interior of } E(f) \textit{ in } \mathbb{R}^n. \\ \mathrm{CV}(f) & = & \{q \in \mathbb{C}^n : (|q_1|, \ldots, |q_n|) \in \breve{E}(f)\} \quad : \quad \textit{convergence domain of } f. \end{array}$$

The power series f is to be said convergent *if $\mathrm{CV}(f) \neq \emptyset$. Let $\mathcal{U}$ be an open domain in $\mathbb{C}^n$ and let $q \in \mathbb{C}^n$. The power series f is to be said convergent on q (resp. over $\mathcal{U}$) if $q \in \mathrm{CV}(f)$ (resp. $\mathcal{U} \subset \mathrm{CV}(f)$). We set*

$$\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!] \quad = \quad \{f \in \mathbb{C}[\![q_1, \ldots, q_n]\!] : \mathrm{CV}(f) \neq \emptyset\}.$$

Let $q \in \mathrm{CV}(f)$. There exists the constants C_f, ρ and $\bar\rho$ such that

$$|q_1| < \bar\rho < \rho, \ldots, |q_n| < \bar\rho < \rho \quad \textit{and} \quad |f_{i_1,\ldots,i_n}|\,\rho_1^{i_1}\ldots\rho_n^{i_n} \leq C_f,$$

for $i_1 \ldots, i_n \geq 0$. The convergence modulus *of f at q is $(C_f, \rho, \bar\rho)$.*

Suppose that $\mathrm{CV}(f) \neq \emptyset$ and let $q \in \mathrm{CV}(f)$. If $(C_f, \rho, \bar\rho)$ is a convergence modulus of f at q then $|f_{i_1,\ldots,i_n} q_1^{i_1}\ldots q_n^{i_n}| \leq C_f(\bar\rho_1/\rho_1)^{i_1}\ldots(\bar\rho_1/\rho_1)^{i_n}$. Hence, at q, the power series f is majored term by term by

$$C_f\prod_{k=0}^{m}\Big(1 - \frac{\bar\rho_k}{\rho_k}\Big)^{-1}. \tag{71}$$

Hence, f is uniformly absolutely convergent in $\{q \in \mathbb{C}^n : \mid q_1 \mid < \bar{\rho}, \ldots, \mid q_n \mid < \bar{\rho}\}$ which is an open domain in $\mathbb{C}^n$. Thus, $\mathrm{CV}(f)$ is an open domain in $\mathbb{C}^n$. Since the partial derivation of order $j_1, \ldots, j_n \geq 0$ of f is estimated by

$$\|D_1^{j_1} \ldots D_n^{j_n} f\| \quad \leq \quad C_f \frac{\partial^{j_1+\ldots+j_n}}{\partial \bar{\rho}^{j_1+\ldots+j_n}} \prod_{k=0}^{m} \left(1 - \frac{\bar{\rho}_k}{\rho_k}\right)^{-1}. \tag{72}$$

Proposition 11 ([25])**.** *We have* $\mathrm{CV}(f) \subset \mathrm{CV}(D_1^{j_1} \ldots D_n^{j_n} f)$.

Let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$. Let $\{A_i\}_{i=0,1}$ be a polysystem defined as follows

$$A_i(q) \quad = \quad \sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}, \tag{73}$$

with, for any $j = 1, \ldots, n$, $A_i^j(q) \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$.

Lemma 10 ([20])**.** *For $i = 0, 1$ and $j = 1, .., n$, one has $A_i \circ q_j = A_i^j(q)$. Thus,*

$$\forall i = 0, 1, \quad A_i(q) \quad = \quad \sum_{j=1}^{n} (A_i \circ q_j) \frac{\partial}{\partial q_j}.$$

Let $(\rho, \bar{\rho}, C_f), \{(\rho, \bar{\rho}, C_i)\}_{i=0,1}$ be respectively the convergence modulus at

$$q \quad \in \quad \mathrm{CV}(f) \bigcap_{\substack{i=0,1 \\ j=1,..,n}} \mathrm{CV}(A_i^j) \tag{74}$$

of f and $\{A_i^j\}_{j=1,..,n}$. Let us consider the following monoid morphisms

$$\mathcal{A}(\epsilon) = \text{identity} \quad \text{and} \quad C(\epsilon) = 1, \tag{75}$$

$$\forall w = v x_i, x_i \in X, v \in X^*, \quad \mathcal{A}(w) = \mathcal{A}(v) A_i \quad \text{and} \quad C(w) = C(v) C_i. \tag{76}$$

Lemma 11 ([19])**.** *For any $w \in X^*$, the derivation $\mathcal{A}(w)$ is continuous over $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$ and, for any $f, g \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$, one has*

$$\mathcal{A}(w) \circ (fg) \quad = \quad \sum_{u,v \in X^*} \langle u \sqcup\!\sqcup v \mid w \rangle \; (\mathcal{A}(u) \circ f)(\mathcal{A}(v) \circ g).$$

These notations are extended, by linearity, to $\mathbb{K}\langle X \rangle$ and we will denote $\mathcal{A}(w) \circ f_{|q}$ the evaluation of $\mathcal{A}(w) \circ f$ at q.

Definition 11 ([19])**.** *Let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$. The generating series of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f is given by*

$$\sigma f \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f \; w \quad \in \quad \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]\langle\!\langle X \rangle\!\rangle.$$

Then the following generating series

$$\sigma f_{|_q} \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|q} \; w \quad \in \quad \mathbb{C}\langle\!\langle X \rangle\!\rangle$$

is called the Fliess' generating series *of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f at q.*

Lemma 12 ([19]). *Let $\{A_i\}_{i=0,1}$ be a polysystem. Then, the map*

$$\sigma : (\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!],.) \quad \longrightarrow \quad (\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\!\langle X\rangle\!\rangle, \sqcup\!\sqcup),$$

is an algebra morphism :

$$\sigma(\nu f + \mu h) = \nu\sigma f + \mu\sigma g \quad \text{and} \quad \sigma(fg) = \sigma f \ \sqcup\!\sqcup\ \sigma g,$$

for any $f, g \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$ and for any $\mu,\nu \in \mathbb{C}$.

Lemma 13 ([20]). *Let $\{A_i\}_{i=0,1}$ be a polysystem and let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$. One has*

$$\begin{array}{rcl}\forall x_i \in X, \quad \sigma(A_i \circ f) = x_i \triangleleft \sigma f & \in & \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\!\langle X\rangle\!\rangle \\ \forall w \in X^*, \quad \sigma(\mathcal{A}(w) \circ f) = w \triangleleft \sigma f & \in & \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\!\langle X\rangle\!\rangle.\end{array}$$

Lemma 14 ([25]). *Let $\tau = \min_{1\le k\le n}\rho_k$ and $r = \max_{1\le k\le n}\bar\rho_k/\rho_k$. We have*

$$\begin{array}{rcl}\|\mathcal{A}(w)\circ f\| & \le & C_f\dfrac{(n+1)}{(1-r)^n}\dfrac{C(w)\,|w|!}{\binom{n+|w|-1}{|w|}}\left[\dfrac{n}{\tau(1-r)^{n+1}}\right]^{|w|} \\ & \le & C_f\dfrac{(n+1)}{(1-r)^n}C(w)\left[\dfrac{n}{\tau(1-r)^{n+1}}\right]^{|w|}|w|!.\end{array}$$

Theorem 8 ([25]). *Let $K = C_f(n+1)(1-r)^{-n}$ and let χ be the real positive function defined over X^* by*

$$\forall i = 0,1, \quad \chi(x_i) \quad = \quad \frac{C_i n}{\tau(1-r)^{(n+1)}}.$$

Then the generating series σf of the polysystem $\{A_i\}_{i=0,1}$ and of the observation $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$ satisfies the $\chi-$growth condition.

It is the same for the Fliess generating series $\sigma f_{|q}$ of the polysystem $\{A_i\}_{i=0,1}$ and of the observation $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$ at q.

3.2 Polysystems and nonlinear differential equation

3.2.1 Nonlinear differential equation

Let us consider the following singular inputs[13]

$$u_0(z) = z^{-1} \quad \text{and} \quad u_1(z) = (1-z)^{-1}, \tag{77}$$

and the following nonlinear dynamical system

$$\left\{\begin{array}{rcl} y(z) & = & f(q(z)), \\ \dot q(z) & = & A_0(q)\, u_0(z) + A_1(q)\, u_1(z), \\ q(z_0) & = & q_0, \end{array}\right. \tag{78}$$

where, the state $q = (q_1,\ldots,q_n)$ belongs to the complex analytic manifold of dimension n, q_0 is the initial state, the observation f belongs to $\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$ and $\{A_i\}_{i=0,1}$ is the polysystem defined on (73).

[13]These singular inputs are not included in the studies of Fliess [19, 20].

Definition 12 (see [27]). *The following power series is called* transport operator *of the polysystem* $\{A_i\}_{i=0,1}$ *and of the observation* f

$$\mathcal{T} \quad := \quad \sum_{w \in X^*} \alpha_{z_0}^z(w)\ \mathcal{A}(w).$$

By a factorization of the monoid by Lyndon words, we have [27]

$$\mathcal{T} = (\alpha_{z_0}^z \otimes \mathcal{A}) \left(\sum_{w \in X^*} w \otimes w \right) = \prod_{l \in \mathcal{L}ynX} \exp[\alpha_{z_0}^z(S_l)\ \mathcal{A}(\check{S}_l)]. \tag{79}$$

Let

$$\sigma f \quad = \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f\ w \tag{80}$$

be the generating series of (78) satisfying the $\chi-$growth condition given on Theorem 8. Let us consider again the Chen generating series $S_{z_0 \rightsquigarrow z}$ given in (61) of the diffferential forms involed in Equation (DE) of Example 1, *i.e.*

$$\omega_0(z) = u_0(z)\ dz \quad \text{and} \quad \omega_1(z) = u_1(z)\ dz \tag{81}$$

This generating series verifies the upper bound conditions given on (67).

3.2.2 Asymptotic behaviour of the successive differentiation of the output via extended Fliess fundamental formula

The Fliess fundamental formula [19] can be then extended as follows

Theorem 9 ([39]). *We have*

$$\begin{aligned} y(z) &= \mathcal{T} \circ f_{|_{q_0}} \\ &= \sum_{w \in X^*} \langle S_{z_0 \rightsquigarrow z} \mid w \rangle \langle \mathcal{A}(w) \circ f_{|_{q_0}} \mid w \rangle \\ &= \langle \sigma f_{|_{q_0}} \parallel S_{z_0 \rightsquigarrow z} \rangle. \end{aligned}$$

Using the factorization indexed by Lyndon words of the Lie exponential series Ł, we deduce some expansions of the output y :

Corollary 2 ([39]). *The output* y *of the nonlinear dynamical system with singular inputs admits the following functional expansions*

$$\begin{aligned} y(z) &= \sum_{w \in X^*} g_w(z)\ \mathcal{A}(w) \circ f_{|q_0}, \\ &= \sum_{k \geq 0} \sum_{n_1, \ldots, n_k \geq 0} g_{x_0^{n_1} x_1 \ldots x_0^{n_k} x_1}(z)\ \mathrm{ad}_{A_0}^{n_1} A_1 \ldots \mathrm{ad}_{A_0}^{n_k} A_1 e^{\log z A_0} \circ f_{|q_0}, \\ &= \prod_{l \in \mathcal{L}ynX} \exp\Big(g_{S_l}(z)\ \mathcal{A}(\check{S}_l) \circ f_{|_{q_0}} \Big), \\ &= \exp\Big(\sum_{w \in X^*} g_w(z)\ \mathcal{A}(\pi_1(w)) \circ f_{|_{q_0}} \Big), \end{aligned}$$

where, for any word w in X^*, g_w belongs to the polylogarithm algebra.

Since $S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ and since $\sigma f_{|_{q_0}}$ and $\mathrm{L}(z_0)^{-1}$ are invariant by $\partial = d/dz$ then, for any integer l, one has

$$\partial^l y(z) = \langle \sigma f_{|_{q_0}} \parallel \partial S_{z_0 \rightsquigarrow z} \rangle = \langle \sigma f_{|_{q_0}} \parallel \partial^l \mathrm{L}(z)\mathrm{L}(z_0)^{-1} \rangle. \tag{82}$$

With the notations of Proposition 3, we get

$$\partial^l y(z) = \langle \sigma f_{|_{q_0}} \parallel [P_l(z)\mathrm{L}(z)]\mathrm{L}(z_0)^{-1} \rangle = \langle \sigma f_{|_{q_0}} \triangleright P_l(z) \parallel \mathrm{L}(z)\mathrm{L}(z_0)^{-1} \rangle. \tag{83}$$

For $z_0 = \varepsilon \to 0^+$, the asymptotic behaviour of $\partial^l y(z)$ at $z = 1$ is deduced then from Proposition 5 as follows

Corollary 3. *For any integer l,we have*

$$\begin{array}{rcl} \partial^l y(1) & \widetilde{\varepsilon \to 0^+} & \langle \sigma f_{|_{q_0}} \triangleright P_l(1-\varepsilon) \parallel e^{-x_1 \log \varepsilon}\, Z_{\sqcup\!\sqcup}\, e^{-x_0 \log \varepsilon} \rangle \\ & = & \displaystyle\sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|_{q_0}} \mid w \rangle \langle P_l(1-\varepsilon)\, e^{-x_1 \log \varepsilon}\, Z_{\sqcup\!\sqcup}\, e^{-x_0 \log \varepsilon} \mid w \rangle. \end{array}$$

Corollary 4. *The differentiation of order $l \in \mathbb{N}$ of the output y of the dynamical system (78) is a $\mathcal{C}$-combination of the elements g belonging to the polylogarithm algebra. If its ordinary Taylor expansion exists then the coefficients of this expansion belong to the algebra of harmonic sums and there exist algorithmically computable coefficients $a_i \in \mathbb{Z}, b_i \in \mathbb{N}$ and c_i belong to the $\mathbb{C}$-algrebra generated by $\mathcal{Z}$ and by the Euler's γ constant, such that*

$$\partial^l y(z) = \sum_{n \geq 0} y_n^{(l)} z^n, \qquad y_n^{(l)} \widetilde{n \to \infty} \sum_{i \geq 0} c_i n^{a_i} \log^{b_i} n.$$

3.3 Differential realization

3.3.1 Differential realization

Definition 13. *The* Lie rank *of a formal power series $S \in \mathbb{K}\langle\langle X \rangle\rangle$ is the dimension of the vector space generated by*

$$\{S \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle\}, \quad \textit{or respectively by} \quad \{\Pi \triangleleft S \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle\}.$$

Definition 14. *Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$ and let us put*

$$\begin{array}{rcl} \mathrm{Ann}(S) & := & \{\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle \mid S \triangleright \Pi = 0\}, \\ \mathrm{Ann}^{\perp}(S) & := & \{Q \in (\mathbb{K}\langle\langle X \rangle\rangle, \sqcup\!\sqcup) \mid Q \triangleright \mathrm{Ann}(S) = 0\}. \end{array}$$

It is immediate that $\mathrm{Ann}^{\perp}(S) \ni S$ and it follows that (see [20, 46])

Lemma 15. *Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$. If S is of finite Lie rank, d, then the dimension of $\mathrm{Ann}^{\perp}(S)$ equals d.*

By Lemma 3, the residuals are derivations for shuffle product. Then,

Lemma 16. *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. Then :*

1. *For any Q_1 and $Q_2 \in \operatorname{Ann}^{\perp}(S)$, one has $Q_1 \sqcup\!\sqcup\, Q_2 \in \operatorname{Ann}^{\perp}(S)$.*
2. *For any $P \in \mathbb{K}\langle X\rangle$ and $Q_1 \in \operatorname{Ann}^{\perp}(S)$, one has $P \triangleleft Q_1 \in \operatorname{Ann}^{\perp}(S)$.*

Definition 15 ([20]). *The formal power series $S \in \mathbb{K}\langle\langle X\rangle\rangle$ is* differentially produced *if there exists*

- *an integer d,*
- *a power series $f \in \mathbb{K}[\![\bar{q}_1,\ldots,\bar{q}_d]\!]$,*
- *a homomorphism $\mathcal{A}$, from X^* into the algebra of differential operators generated by*

$$\mathcal{A}(x_i) \quad = \quad \sum_{j=1}^{d} A_i^j(\bar{q}_1,\ldots,\bar{q}_d)\frac{\partial}{\partial \bar{q}_j},$$

  where, for any $j = 1,\ldots,d$, $A_i^j(\bar{q}_1,\ldots,\bar{q}_d)$ belongs to $\mathbb{K}[\![\bar{q}_1,\ldots,\bar{q}_d]\!]$ such that, for any $w \in X^*$, one has

$$\langle S \mid w\rangle \quad = \quad \mathcal{A}(w)\circ f_{|_0}.$$

The couple $(\mathcal{A}, f)$ is called differential representation *of S of dimension d.*

Proposition 12 ([46]). *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. If S is differentially produced then it verifies the growth condition and its Lie rank is finite.*

Proof. Let $(\mathcal{A}, f)$ be a differential representation of S of dimension d. Then, by the notations of Definition 11, we get

$$\sigma f_{|_0} = S = \sum_{w\in X^*} (\mathcal{A}(w)\circ f)_{|_0}\, w.$$

For any $j = 1,..,d$, we put

$$T_j = \sum_{w\in X^*} \frac{\partial(\mathcal{A}(w)\circ f)}{\partial \bar{q}_j}\, w \quad\Longleftrightarrow\quad \forall w \in X^*,\quad \langle T_j \mid w\rangle = \frac{\partial(\mathcal{A}(w)\circ f)}{\partial \bar{q}_j}.$$

Firstly, by Theorem 8, the generating series σf verifies the growth condition. Secondly, for any $\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ and for any $w \in X^*$, one has

$$\langle \sigma f \triangleright \Pi \mid w\rangle = \langle \sigma f \mid \Pi w\rangle = \mathcal{A}(\Pi w)\circ f = \mathcal{A}(\Pi)\circ(\mathcal{A}(w)\circ f).$$

Since $\mathcal{A}(\Pi)$ is a derivation over $\mathbb{K}[\![\bar{q}_1,\ldots,\bar{q}_d]\!]$:

$$\begin{aligned} \mathcal{A}(\Pi) \quad &= \quad \sum_{j=1}^{d}(\mathcal{A}(\Pi)\circ \bar{q}_j)\frac{\partial}{\partial \bar{q}_j}, \\ \Rightarrow \quad \mathcal{A}(\Pi)\circ(\mathcal{A}(w)\circ f) \quad &= \quad \sum_{j=1}^{d}(\mathcal{A}(\Pi)\circ \bar{q}_j)\frac{\partial(\mathcal{A}(w)\circ f)}{\partial \bar{q}_j} \end{aligned}$$

then we deduce that

$$\begin{aligned}\forall w \in X^*, \quad \langle \sigma f \triangleright \Pi \mid w\rangle &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j)\langle T_j \mid w\rangle, \\ \iff \quad \sigma f \triangleright \Pi &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j)\, T_j\end{aligned}$$

That means $\sigma f \triangleright \Pi$ is $\mathbb{K}$-linear combination of $\{T_j\}_{j=1,..,d}$ and the dimension of $\mathrm{span}\{\sigma f \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}$ is less than or equal to d. ☐

3.3.2 Fliess' local realization theorem

Proposition 13 ([46]). *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ such that its Lie rank equals d. Then there exists a basis $S_1, \ldots, S_d \in \mathbb{K}\langle\langle X\rangle\rangle$ of $(\mathrm{Ann}^{\perp}(S), ⧢) \cong (\mathbb{K}[\![S_1, \ldots, S_d]\!], ⧢)$ such that the S_i's are proper and for any $R \in \mathrm{Ann}^{\perp}(S)$, one has*

$$R = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \ldots i_d!} S_1^{⧢ i_1} ⧢ \ldots ⧢ S_d^{⧢ i_d},$$

where the coefficients $\{r_{i_1,\ldots,i_d}\}_{i_1,\ldots,i_d \geq 0}$ belong to $\mathbb{K}$ and $r_{0,\ldots,0} = \langle R \mid \epsilon\rangle$.

Proof. By Lemma 15, a such basis exists. More precisely, since the Lie rank of S is d then there exists proper Lie polynomials $P_1, \ldots, P_d \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ such that $S \triangleright P_1, \ldots, S \triangleright P_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ are $\mathbb{K}$-linearly independent. By duality, their exists $S_1, \ldots, S_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ such that

$$\forall i, j = 1, ..d, \quad \langle S_i \mid P_j\rangle = \delta_i^j, \quad \text{and} \quad R = \prod_{i=1}^{d} \exp(S_i\ P_i).$$

Expending this product, one obtains the expected expression for the coefficients $\{r_{i_1,\ldots,i_d}\}_{i_1,\ldots,i_d \geq 0}$ (via Poincaré-Birkhoff-Witt theorem, see [46])

$$r_{i_1,\ldots,i_d} = \langle R \mid P_1^{i_1} \ldots P_d^{i_d}\rangle.$$

Hence, $(\mathrm{Ann}^{\perp}(S), ⧢)$ is generated by $S_1, \ldots, S_d$. ☐

With the notations of Proposition 13, one has respectively

Corollary 5. *If $S \in \mathbb{K}[S_1, \ldots, S_d]$ then, for any $i = 0, 1$ and for any $j = 1, .., d$, one has $x_i \triangleleft S \in \mathrm{Ann}^{\perp}(S) = \mathbb{K}[S_1, \ldots, S_d]$.*

Corollary 6. *The power series S verifies the growth condition if and only if, for any $i = 1, ..d$, S_i also verifies the growth condition.*

Proof. Assume their exists $j \in [1, .., d]$ such that S_j does not verify the growth condition. Since $S \in \mathrm{Ann}^{\perp}(S)$ then using the decomposition of S on $S_1, \ldots, S_d$, one obtains a contradiction with the fact that S verifies the growth condition.

Conservely, using Proposition 8, we get the expected results. ☐

Theorem 10 (Fliess' local realization theorem, [20])**.** *The formal power series $S \in \mathbb{K}\langle\langle X \rangle\rangle$ is differentially produced if and only if its Lie rank is finite and if it verifies the χ-growth condition.*

Proof. By Proposition 12, one get a direct proof.

Conversely, since the Lie rank of S equals d then by Proposition 13, for any $j = 1, .., d$, by putting $\sigma f_{|0} = S$ and $\sigma \bar{q}_i = S_i$,

1. we choose the observation f as follows

$$f(\bar{q}_1, \ldots, \bar{q}_d) \quad = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \ldots i_d!} \bar{q}_1^{i_1} \ldots \bar{q}_d^{i_d} \quad \in \quad \mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!],$$

   such that

$$\sigma f_{|0}(\bar{q}_1, \ldots, \bar{q}_d) \quad = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \ldots i_d!} (\sigma \bar{q}_1)^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup (\sigma \bar{q}_d)^{\sqcup\!\sqcup\, i_d},$$

2. it follows that, for $i = 0, 1$ and for $j = 1, .., d$, the residuals $x_i \triangleleft \sigma \bar{q}_j$ belongs to $\mathrm{Ann}^{\perp}(\sigma f_{|0})$ (see also Lemma 16),

3. since σf verifies the χ-growth condition then, by Corollary 6, the generating series $\sigma \bar{q}_j$ and $x_i \triangleleft \sigma \bar{q}_j$ (for $i = 0, 1$ and for $j = 1, .., d$) verify also the growth condition. We then take (see Lemma 13)

$$\forall i = 0, 1, \quad \forall j = 1, .., d, \quad \sigma A_j^i(\bar{q}_1, \ldots, \bar{q}_d) = x_i \triangleleft \sigma \bar{q}_j,$$

   by expressing σA_j^i on the basis $\{\sigma \bar{q}_i\}_{i=1,..,d}$ of $\mathrm{Ann}^{\perp}(\sigma f_{|0})$,

4. the homomorphism $\mathcal{A}$ is then determined as follows

$$\forall i = 0, 1, \quad \mathcal{A}(x_i) \quad = \quad \sum_{j=0}^{d} A_j^i(\bar{q}_1, \ldots, \bar{q}_d) \frac{\partial}{\partial \bar{q}_j},$$

   where (see Lemma 10),

$$\forall i = 0, 1, \quad \forall j = 1, .., d, \quad A_j^i(\bar{q}_1, \ldots, \bar{q}_d) = \mathcal{A}(x_i) \circ \bar{q}_j.$$

Thus, $(\mathcal{A}, f)$ provides a differential representation[14] of dimension d of S. □

Moreover, one also has the following

Theorem 11 ([20])**.** *Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$ supposed to be a differentially produced formal power series.*

If $(\mathcal{A}, f)$ and $(\mathcal{A}', f')$ are two differential representations of dimension n of S then there exists a continuous and convergent automorphism h of $\mathbb{K}$ such that

$$\forall w \in X^*, \forall g \in \mathbb{K}, \quad h(\mathcal{A}(w) \circ g) = \mathcal{A}'(w) \circ (h(g)) \quad \textit{and} \quad f' = h(f).$$

[14] In [20, 46], the reader can found the discussion on the *minimal* differential representation.

Since any rational power series (resp. polynomial), verifies the growth condition and its Lie rank is less or equal to its Hankel rank which is finite [20] then

Corollary 7. *Any rational power series and any polynomial over X with coefficients in $\mathbb{K}$ are differentially produced.*

Remark 1.

1. Note that, by Corollary 5, if S is polynomial over X then for any $j = 1, .., d$, S_j is polynomial over X. Therefore, for any $i = 0, 1$ and for any $j = 1, .., d$, $x_i \triangleleft S$ is also polynomial over X. In this case, let $(\mathcal{A}, f)$ be a differential representation of P of dimension d. Then f and $\{A_j^i\}_{j=1,..,d}^{i=0,1}$ are obviously polynomial on $\bar{q}_1, \ldots, \bar{q}_d$ and the Lie algebra generated by $\{\mathcal{A}(x_i)\}^{i=0,1}$ is nilpotent.

2. Note also that, by Theorem 6, if S is rational over X of linear representation (λ, μ, η) then the observation $f(q_1, \ldots, q_n)$ equals $\lambda_1 q_1 + \ldots + \lambda_n q_n$ and the polysystem $\{\mathcal{A}(x)\}_{x\in X}$ is obtained by putting
$$\forall x_i \in X, \quad \mathcal{A}(x_i) \quad = \quad \sum_{j=1}^{n} (\mu(x_i))_j^i \frac{\partial}{\partial q_j}.$$
This gives a *linear* differential representation which is not necessarily of minimal dimension [20].

3. Assume $S \in \mathbb{K}\epsilon \oplus x_0\mathbb{K}\langle\langle X\rangle\rangle x_1$ and S is a differentially produced. If there exists a basis $S_1, \ldots, S_d$ of $(\mathrm{Ann}^{\perp}(S), ш) \cong (x_0\mathbb{K}\langle\langle X\rangle\rangle x_1, ш)$ such that
$$S \quad = \quad \sum_{i_1,\ldots,i_d \geq 0} r_{i_1,\ldots,i_n} \frac{S_1^{ш\, i_1}}{i_1!} ш \ldots ш \frac{S_d^{ш\, i_d}}{i_d!} \;\in\; \mathbb{K}[S_1, \ldots, S_d]. \quad (84)$$
For $i = 1, .., d$, we put $\Sigma_i = \Pi_Y S_i$ and let
$$\Sigma \quad = \quad \sum_{i_1,\ldots,i_d \geq 0} r_{i_1,\ldots,i_n} \frac{\Sigma_1^{ш\!\!\!+\, i_1}}{i_1!} ш\!\!\!+ \ldots ш\!\!\!+ \frac{\Sigma_d^{ш\!\!\!+\, i_d}}{i_d!}. \quad (85)$$
It is a generalization of a Radford's theorem because [29, 30]:
   - If $S \in \mathbb{Q}\langle X\rangle$ then (84) and (85) are decompositions on Radford bases.
   - If S is rational then these are *noncommutative partial decompositions.* In this case, one has in general
   $$\Pi_Y S \quad \neq \quad \Sigma \quad (86)$$
   but
   $$\zeta(S_i) \quad = \quad \zeta(\Sigma_i) \quad (87)$$
   and
   $$\zeta(S) \quad = \quad \zeta(\Sigma) \quad = \quad \sum_{i_1,\ldots,i_d \geq 0} r_{i_1,\ldots,i_n} \frac{\zeta(S_1)^{i_1}}{i_1!} \ldots \frac{\zeta(S_d)^{i_d}}{i_d!}. \quad (88)$$
   Thus, these yield also identity on polyzêtas at arbitrary weight [37].

4 The group of associators and the structure of polyzêtas

4.1 Generalized Euler constants and three regularizations of divergent polyzêtas

4.1.1 Regularizations of divergent polyzêtas

Theorem 12 ([33]). *Let $\zeta_{\text{⧢}} : (\mathbb{Q}\langle\langle Y\rangle\rangle, \text{⧢}) \to (\mathbb{R}, .)$ be the morphism verifying the following properties*

- *for $u, v \in Y^*, \zeta_{\text{⧢}}(u \text{⧢} v) = \zeta_{\text{⧢}}(u)\zeta_{\text{⧢}}(v)$,*
- *for all convergent word $w \in Y^* - y_1Y^*, \zeta_{\text{⧢}}(w) = \zeta(w)$,*
- *$\zeta_{\text{⧢}}(y_1) = 0$.*

Then

$$\sum_{w\in X^*} \zeta_{\text{⧢}}(w)\, w \quad = \quad Z_{\text{⧢}}.$$

Corollary 8 ([33]). *For any $w \in X^*, \zeta_{\text{⧢}}(w)$ belongs to the algebra $\mathcal{Z}$.*

Theorem 13 ([33]). *Let $\zeta_{\sqcup\!\sqcup} : (\mathbb{Q}\langle\langle X\rangle\rangle, \sqcup\!\sqcup) \to (\mathbb{R}, .)$ be the morphism verifying the following properties*

- *for $u, v \in X^*, \zeta_{\sqcup\!\sqcup}(u \sqcup\!\sqcup v) = \zeta_{\sqcup\!\sqcup}(u)\zeta_{\sqcup\!\sqcup}(v)$,*
- *for all convergent word $w \in x_0X^*x_1, \zeta_{\sqcup\!\sqcup}(w) = \zeta(w)$,*
- *$\zeta_{\sqcup\!\sqcup}(x_0) = \zeta_{\sqcup\!\sqcup}(x_1) = 0$.*

Then

$$\sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(w)\, w \quad = \quad Z_{\sqcup\!\sqcup}.$$

Corollary 9 ([33]). *For any $w \in Y^*, \zeta_{\sqcup\!\sqcup}(w)$ belongs to the algebra $\mathcal{Z}$.*

Definition 16. *For any $w \in Y^*$, let γ_w be the constant part[15] of the asymptotic expasion, on the comparison scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$, of $\mathrm{H}_w(n)$.*

Let Z_γ be the noncommutative generating series of $\{\gamma_w\}_{w\in Y^*}$:

$$Z_\gamma \quad := \quad \sum_{w\in Y^*} \gamma_w\, w.$$

Definition 17. *We set*

$$B(y_1) := \exp\left(-\sum_{k\geq 1} \gamma_{y_k} \frac{(-y_1)^k}{k}\right) \quad \text{and} \quad B'(y_1) := e^{-\gamma y_1} B(y_1).$$

[15] *i.e.* γ_w is the Euler-Mac Laurin of $\mathrm{H}_w(n)$.

The power series $B'(y_1)$ corresponds in fact to the mould[16] Mono in [17] and to the $\Phi_{\rm corr}$ in [43] (see also [4, 8]). While the power series $B(y_1)$ corresponds to the Gamma Euler function with its product expansion,

$$B(y_1) = \Gamma(y_1+1), \qquad \frac{1}{\Gamma(y_1+1)} = e^{\gamma y_1} \prod_{n\geq 1} \left(1+\frac{y_1}{n}\right) e^{-\gamma/n}. \tag{89}$$

Lemma 17. *Let $b_{n,k}(t_1,\ldots,t_{n-k+1})$ be the (exponential) partial Bell polynomials in the variables $\{t_l\}_{l\geq 1}$ given by the exponential generating series*

$$\exp\left(u\sum_{l=0}^{\infty} t_l \frac{v^l}{l!}\right) \quad = \quad \sum_{n,k=0}^{\infty} b_{n,k}(t_1,\ldots,t_{n-k+1}) \frac{v^n u^k}{n!}.$$

For any $m \geq 1$, let $t_m = (-1)^m (m-1)! \gamma_{y_m}$. Then

$$B(y_1) \quad = \quad 1 + \sum_{n\geq 1} \left(\sum_{k=1}^{n} b_{n,k}(\gamma, -\zeta(2), 2\zeta(3), \ldots) \right) \frac{(-y_1)^n}{n!}.$$

Since the ordinary generating series of the finite parts of coefficients of $\mathrm{Const}(N)$ is nothing else but the power series $B(y_1)$, taking the constant part on either side of $\mathrm{H}(N) \underset{N\to\infty}{\widetilde{\ }} \mathrm{Const}(N)\Pi_Y Z_{\sqcup\!\sqcup}$ (see Proposition 4), yields

Theorem 14 ([39]). *We have $Z_\gamma = B(y_1)\Pi_Y Z_{\sqcup\!\sqcup}$.*

Therefore, identifying the coefficients of $y_1^k w$ on either side using the identity[17] (see [33])

$$\forall u \in X^* x_1, \quad x_1^k x_0 u \quad = \quad \sum_{l=0}^{k} x_1^l \sqcup\!\sqcup (x_0[(-x_1)^{k-l} \sqcup\!\sqcup u]). \tag{90}$$

Applying the morphism $\zeta_{\sqcup\!\sqcup}$ given in Theorem 13, we get [33]

$$\forall u \in X^* x_1, \quad \zeta_{\sqcup\!\sqcup}(x_1^k x_0 u) \quad = \quad \zeta(x_0[(-x_1)^k \sqcup\!\sqcup u]). \tag{91}$$

Corollary 10 ([39]). *For $w \in x_0 X^* x_1$,* i.e. *$w = x_0 u$ and $\Pi_Y w \in Y^* - y_1 Y^*$, and for $k \geq 0$, the constant $\gamma_{\sqcup\!\sqcup}(x_1^k w)$ associated to the divergent polyzêta $\zeta(x_1^k w)$ is a polynomial of degree k in γ and with coefficients in $\mathcal{Z}$:*

$$\gamma_{x_1^k w} \quad = \quad \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \sqcup\!\sqcup u])}{i!} \left(\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots) \right).$$

[16] The readers can see why we have introduced the power series $\mathrm{Mono}(z)$ in Proposition 4.

[17] By the Convolution Theorem [28], this is equivalent to

$$\begin{aligned} \forall u \in X^*, \quad \alpha_0^z(x_1^k x_0 u) \quad &= \quad \int_0^z \frac{[\log(1-s) - \log(1-z)]^k}{k!} \alpha_0^s(u) \frac{ds}{s} \\ &= \quad \sum_{l=0}^{k} \frac{[-\log(1-z)]^l}{l!} \int_0^z \frac{\log^{k-l}(1-s)}{(k-l)!} \alpha_0^s(u) \frac{ds}{s}. \end{aligned}$$

This theorem induces *de facto* the algebra morphism of regularization to 0 with respect to the shuffle product, as shown the Theorem 13.

Moreover, for $l = 0, .., k$, the coefficient of γ^l is of weight $\mid w \mid +k - l$.

In particular, for $s > 1$, the constant $\gamma_{y_1 y_s}$ associated to $\zeta(y_1 y_s)$ is linear in γ and with coefficients in $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0<i\leq(s-1)/2}$.

Corollary 11 ([39]). *The constant $\gamma_{x_1^k}$ associated to the divergent polyzêta $\zeta(x_1^k)$ is a polynomial of degree k in γ with coefficients in $\mathbb{Q}[\zeta(2), \zeta(2i+1)]_{0<i\leq(k-1)/2}$:*

$$\gamma_{x_1^k} = \sum_{\substack{s_1,\ldots,s_k\geq 0\\ s_1+\ldots+ks_k=k+1}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Moreover, for $l = 0, .., k$, the coefficient of γ^l is of weight $k - l$.

We thereby obtain the following algebra morphism for the regularization to γ with respect to the quasi-shuffle product *independently* to the regularizations with respect to the shuffle product (in [4, 8, 21, 48], the authors suggest the *simultaneous* regularization to T and then to set $T = 0$) :

Theorem 15 ([39]). *The mapping $\gamma_\bullet$ realizes the morphism from $(\mathbb{Q}\langle\langle Y\rangle\rangle, \uplus)$ to $(\mathbb{R}, .)$ verifying the following properties*

- *for any word $u, v \in Y^*, \gamma_{u \uplus v} = \gamma_u \gamma_v$,*
- *for any convergent word $w \in Y^* - y_1 Y^*, \gamma_w = \zeta(w)$*
- *$\gamma_{y_1} = \gamma$.*

Then $Z_\gamma = e^{\gamma y_1} Z_{\uplus}$.

4.1.2 Identities of noncommutative generating series of polyzêtas

Corollary 12. *With the notations of Definition 17, we have*

$$\begin{aligned} Z_\gamma = B(y_1)\Pi_Y Z_{\sqcup\!\sqcup} &\iff Z_{\uplus} = B'(y_1)\Pi_Y Z_{\sqcup\!\sqcup}, \\ \Pi_Y Z_{\sqcup\!\sqcup} = B^{-1}(x_1) Z_\gamma &\iff Z_{\sqcup\!\sqcup} = B'^{-1}(x_1)\Pi_X Z_{\uplus}. \end{aligned}$$

Roughly speaking, for the quasi-shuffle product, the regularization to γ is "equivalent" to the regularization to 0.

Note also that the constant $\gamma_{y_1} = \gamma$ is obtained as the finite part of the asymptotic expansion of $\mathrm{H}_1(n)$ in the comparison scale $\{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$.

In the same way, since n and $\mathrm{H}_1(n)$ are algebraically independent, as arithmetical functions (see Proposition 1), then $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ constitutes a new comparison scale for asymptotic expansions. Hence, the constants $\zeta_{\sqcup\!\sqcup}(x_1) = 0$ and $\zeta_{\uplus}(y_1) = 0$ can be interpreted as the finite part of the asymptotic expansions of $\mathrm{Li}_1(z)$ and $\mathrm{H}_1(n)$ in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ respectively.

Definition 18 ([33]). *We put*

$$C_1 = \mathbb{Q}\epsilon \oplus x_0\mathbb{Q}\langle X\rangle x_1 \quad and \quad C_2 = \mathbb{Q}\epsilon \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle.$$

Lemma 18 ([32, 33]). *We get* $(C_1, \shuffle) \cong (C_2, \stuffle)$.

Using a Radford's theorem [45] and its generalization over Y [23], we have

Proposition 14 ([32, 33]).

$$
\begin{array}{rcccl}
(\mathbb{Q}\langle X\rangle, \shuffle) & \cong & (\mathbb{Q}[\mathcal{L}ynX], \shuffle) & = & C_1[x_0, x_1],\\
(\mathbb{Q}\langle Y\rangle, \stuffle) & \cong & (\mathbb{Q}[\mathcal{L}ynY], \stuffle) & = & C_2[y_1],
\end{array}
$$

This insures the effective way to get the finite part of the asymptotic expansions, in the comparison scales $\{(1-z)^a \log(1-z)^b\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, of $\{\mathrm{Li}_w(z)\}_{w\in Y^*}$ and $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ respectively.

Proposition 15 ([32, 33]). *The restrictions of the morphisms* $\zeta_{\shuffle}$ *and* $\zeta_{\stuffle}$ *over the shuffle algebras* $(C_1, \shuffle)$ *and* $(C_2, \stuffle)$ *respectively coincide with the* surjective *algebra morphism*

$$
\begin{array}{rcl}
\zeta \quad : \quad \begin{matrix}(C_2, \stuffle)\\ (C_1, \shuffle)\end{matrix} & \longrightarrow & (\mathbb{R}, .)\\
\begin{matrix} y_{r_1}\ldots y_{r_k}\\ x_0x_1^{r_1-1}\ldots x_0x_1^{r_k-1}\end{matrix} & \longmapsto & \displaystyle\sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{r_1}\ldots n_k^{r_k}},
\end{array}
$$

In Section 4.3.2 we will give the complete description of the fernel $\ker\zeta$.

With the double regularization[18] to zero [4, 8, 33, 43], the Drindfel'd associator Φ_{KZ} corresponds then to $Z_{\shuffle}$ (obtained with only convergent polyzêtas) as being the unique group-like element satisfying [35, 32]

$$
\langle Z_{\shuffle} \mid x_0\rangle = \langle Z_{\shuffle} \mid x_1\rangle = 0 \quad \text{and} \quad \forall x \in x_0X^*x_1, \quad \langle Z_{\shuffle} \mid w\rangle = \zeta(w). \tag{92}
$$

As consequence of Proposition 2, one has

Proposition 16 ([38]).

$$
\begin{aligned}
\log Z_{\shuffle} &= \sum_{w\in X^*} \zeta_{\shuffle}(w)\, \pi_1(w),\\
&= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in X^*-\{\epsilon\}} \zeta_{\shuffle}(u_1 \shuffle \ldots \shuffle u_k)\, u_1\ldots u_k.
\end{aligned}
$$

The associator Φ_{KZ} can be also graded in the adjoint basis as follows

Proposition 17 ([38]). *For any* $l \in \mathbb{N}$ *and* $P \in \mathbb{C}\langle X\rangle$, *let* $\circ$ *denotes the composite operation defined by* $x_1x_0^l \circ P = x_1(x_0^l \shuffle P)$. *Then*

$$
Z_{\shuffle} \;=\; \sum_{k\geq 0} \sum_{l_1,\cdots,l_k\geq 0} \zeta_{\shuffle}(x_1x_0^{l_1} \circ \cdots \circ x_1x_0^{l_k}) \prod_{i=0}^{k} \mathrm{ad}_{x_0}^{l_i} x_1,
$$

where $\mathrm{ad}_{x_0}^l x_1$ *is iterated Lie bracket* $\mathrm{ad}_{x_0}^l x_1 = [x_0, \mathrm{ad}_{x_0}^{l-1} x_1]$ *and* $\mathrm{ad}_{x_0}^0 x_1 = x_1$.

[18]This double regularization is deduced from of the noncommutative generating series $Z_{\shuffle}$ and $Z_{\stuffle}$ on the definitions 1 and 2 (see the theorems 12 and 13).

Using the following expansion [6]

$$\mathrm{ad}_{x_0}^n x_1 = \sum_{i=0}^{n} \binom{i}{n} x_0^{n-i} x_1 x_0^i, \tag{93}$$

one gets then, via the regularization process of Theorem 13, the expression of the Drindfel'd associator Φ_{KZ} given by Lê and Murakami [22].

4.2 Action of the differential Galois group of polylogarithms on their asymptotic expansions

4.2.1 Group of associators theorem

Let A a be a commutative $\mathbb{Q}$-algebra.

Since the polyzêtas satisfy (46), then by the Friedrichs criterion we can state the following

Definition 19. *Let $dm(A)$ be the set of $\Phi \in A\langle\langle X \rangle\rangle$ such that*[19]

$$\langle \Phi \mid \epsilon \rangle = 1, \quad \langle \Phi \mid x_0 \rangle = \langle \Phi \mid x_1 \rangle = 0, \quad \Delta_{\sqcup\!\sqcup} \Phi = \Phi \otimes \Phi$$

and such that, for $\Psi = B'(y_1)\Pi_Y \Phi \in A\langle\langle Y \rangle\rangle$ then[20] $\Delta_{\boxplus}\Psi = \Psi \otimes \Psi$.

Proposition 18 ([38])**.** *If $G(z)$ and $H(z)$ are exponential solutions of (DE) then there exsists a Lie series $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle$ such that $G(z) = H(z)\exp(C)$.*

Proof. Since $H(z)H(z)^{-1} = 1$ then by differentiating, we have

$$d[H(z)]H(z)^{-1} = -H(z)d[H(z)^{-1}].$$

Therefore if $H(z)$ is solution of Drinfel'd equation then

$$\begin{aligned} d[H(z)^{-1}] &= -H(z)^{-1}[dH(z)]H(z)^{-1} \\ &= -H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)], \\ d[H(z)^{-1}G(z)] &= H(z)^{-1}[dG(z)] + [dH(z)^{-1}]G(z) \\ &= H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z) \\ &- H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z). \end{aligned}$$

By simplification, we deduce then $H(z)^{-1}G(z)$ is a constant formal power series. Since the inverse and the product of group like elements is group like then we get the expected result. □

The differential $\mathcal{C}$-module $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is the universal Picard-Vessiot extension of every linear differential equations, with coefficients in $\mathcal{C}$ and admitting $\{0, 1, \infty\}$ as regular singularities. The universal differential Galois group, noted by $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$, is the set of differential $\mathcal{C}$-automorphisms of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (*i.e* the

[19] $\Delta_{\sqcup\!\sqcup}$ denotes the coproduct of the shuflle product.

[20] $\Delta_{\boxplus}$ denotes the coproduct of the quasi-shuflle product.

automorphisms of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ that let $\mathcal{C}$ point-wise fixed and that commute with derivation). The action of an automorphism of $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ can be determined by its action on Li_w, for $w \in X^*$. It can be resumed as its action on the noncommutative generating series Ł [38] :

Let $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$. Then

$$\sum_{w\in X^*} \sigma \,\mathrm{Li}_w \; w \quad = \quad \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\sigma \,\mathrm{Li}_{\check{S}_l} \, S_l}. \tag{94}$$

Since $d\sigma \,\mathrm{Li}_{x_i} = \sigma d \,\mathrm{Li}_{x_i} = \omega_i$ then by integrating the two memmbers, we have $\sigma \,\mathrm{Li}_{x_i} = \mathrm{Li}_{x_i} + c_{x_i}$, where c_{x_i} is a constant of integration. More generally, for any Lyndon word $l = x_i l_1^{i_1} \cdots l_k^{i_k}$ with $l_1 > \cdots > l_k$, one has

$$\sigma \,\mathrm{Li}_{\check{S}_l} \quad = \quad \int \omega_{x_i} \frac{\sigma \,\mathrm{Li}^{i_1}_{\check{S}_{l_1}}}{i_1!} \cdots \frac{\sigma \,\mathrm{Li}^{i_k}_{\check{S}_{l_k}}}{i_k!} + c_{\check{S}_l}, \tag{95}$$

where $c_{\check{S}_l}$ is a constant of integration. For example,

$$\sigma \,\mathrm{Li}_{x_0x_1} \quad = \quad \mathrm{Li}_{x_0x_1} + c_{x_1} \mathrm{Li}_{x_0} + c_{x_0x_1}, \tag{96}$$

$$\sigma \,\mathrm{Li}_{x_0^2x_1} \quad = \quad \mathrm{Li}_{x_0^2x_1} + \frac{c_{x_1}}{2} \mathrm{Li}^2_{x_0} + c_{x_0x_1} \mathrm{Li}_{x_0} + c_{x_0^2x_1}, \tag{97}$$

$$\sigma \,\mathrm{Li}_{x_0x_1^2} \quad = \quad \mathrm{Li}_{x_0x_1^2} + c_{x_1} \mathrm{Li}_{x_0x_1} + \frac{c^2_{x_1}}{2} \mathrm{Li}_{x_0} + c_{x_0x_1^2}. \tag{98}$$

Consequently,

$$\sum_{w\in X^*} \sigma \,\mathrm{Li}_w \; w = \text{Ł} e^{C_\sigma} \quad \text{where} \quad e^{C_\sigma} := \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{c_{\check{S}_l} S_l}. \tag{99}$$

The action of $\sigma \in \mathrm{Gal}(\mathrm{LI}_{\mathcal{C}})$ over $\{\mathrm{Li}_w\}_{w\in X^*}$ is then equivalent to the action of the Lie exponential $e^{C_\sigma} \in \mathrm{Gal}(DE)$ over the exponential solution Ł. So,

Theorem 16 ([38])**.** *We have* $\mathrm{Gal}(\mathrm{LI}_{\mathcal{C}}) = \{e^C \;\mid\; C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle\}$.

Typically, since $\text{Ł}(z_0)^{-1}$ is group-like then $S_{z_0\rightsquigarrow z} = \text{Ł}(z)\text{Ł}(z_0)^{-1}$ is an other solution of (44) as already saw in (63).

Theorem 17 (Group of associators theorem)**.** *Let* $\Phi \in A\langle\langle X\rangle\rangle$ *and* $\Psi \in A\langle\langle Y\rangle\rangle$ *be group-like elements, for the coproducts* $\Delta_{\sqcup\!\sqcup}$ *and* $\Delta_{\uplus}$ *respectively, such that*

$$\Psi \quad = \quad B(y_1)\Pi_Y \Phi.$$

Then, there exists an unique $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ *such that*

$$\Phi = Z_{\sqcup\!\sqcup} e^C \quad \textit{and} \quad \Psi = B(y_1)\Pi_Y(Z_{\sqcup\!\sqcup} e^C).$$

Proof. If $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ then $\mathrm{L}' = \mathrm{L}e^C$ is group-like, for the coproduct $\Delta_{\sqcup\!\sqcup}$, and $e^C \in \mathrm{Gal}(DE)$. Let H' be the noncommutative generating series of the Taylor coefficients, belonging to the harmonic algebra, of $\{(1-z)^{-1}\langle \mathrm{L}' \mid w\rangle\}_{w\in Y^*}$. Then $\mathrm{H}'(N)$ is also group-like, for the coproduct $\Delta_{⧢}$.

By the asymptotic expansion of L at $z = 1$ [36, 32], we have

$$\mathrm{L}'(z) \quad \widetilde{_{\varepsilon\to 1}} \quad e^{-x_1 \log(1-z)} Z_{\sqcup\!\sqcup} e^C.$$

We put then $\Phi := Z_{\sqcup\!\sqcup} e^C$ and we deduce that

$$\frac{\mathrm{L}'(z)}{1-z} \widetilde{_{z\to 1}} \mathrm{Mono}(z)\Phi \quad \text{and} \quad \mathrm{H}'(N) \widetilde{_{N\to\infty}} \mathrm{Const}(N)\Pi_Y\Phi,$$

where the expressions of $\mathrm{Mono}(z)$ and $\mathrm{Const}(N)$ are given on (59) and (60) respectively. Let κ_w be the constant part of $\mathrm{H}'_w(N)$. Then

$$\sum_{w\in Y^*} \kappa_w\, w \quad = \quad B(y_1)\Pi_Y\Phi.$$

We put then $\Psi := B(y_1)\Pi_Y\Phi$ (and also $\Psi' := B'(y_1)\Pi_Y\Phi$). □

Corollary 13. *We have*

$$dm(A) = \{Z_{\sqcup\!\sqcup} e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle \quad \text{and } \langle e^C \mid \epsilon\rangle = 1, \langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0\}.$$

Proof. On the one hand, $\langle \Phi \mid x_0\rangle = \langle Z_{\sqcup\!\sqcup} \mid x_0\rangle = 0$, $\langle \Phi \mid x_1\rangle = \langle Z_{\sqcup\!\sqcup} \mid x_1\rangle = 0$ and on the other, $\langle \Phi \mid \epsilon\rangle = \langle Z_{\sqcup\!\sqcup} \mid \epsilon\rangle = 1$, the result follows. □

With the notation of Corollary 12,

Corollary 14. *For any associator $\Phi = Z_{\sqcup\!\sqcup} e^C \in dm(A)$, let $\Psi = B(y_1)\Pi_Y\Phi$ and let $\Psi' = B'(y_1)\Pi_Y\Phi$. Then*

$$\Psi = B(y_1)\Pi_Y\Phi \quad \Longleftrightarrow \quad \Psi' = B'(y_1)\Pi_Y\Phi.$$

Proof. Since Ψ is group like and since $\langle \Phi \mid x_1\rangle = \langle \Psi' \mid y_1\rangle = 0$ and $\langle \Psi \mid y_1\rangle = \gamma$ then, using the factorization by Lyndon words, we get the expected result. □

Lemma 19. *Let $\Phi = Z_{\sqcup\!\sqcup} e^C \in dm(A)$ and let $\Psi = B(y_1)\Pi_Y(Z_{\sqcup\!\sqcup} e^C)$.*

The local coordinates (of second kind) of Φ (resp. Ψ), in the Lyndon-PBW basis, are polynomials on the generators $\{\zeta_{\sqcup\!\sqcup}(l)\}_{l\in\mathcal{L}ynX}$ (resp. $\{\zeta_{⧢}(l)\}_{l\in\mathcal{L}ynY}$) of $\mathcal{Z}$ (resp. $\mathcal{Z}'$). While C describes $\mathcal{L}ie_A\langle\langle X\rangle\rangle$, these local coordinates describe the algebra $A[\{\zeta_{\sqcup\!\sqcup}(l)\}_{l\in\mathcal{L}ynX}]$ (resp. $A[\{\zeta_{⧢}(l)\}_{l\in\mathcal{L}ynY}]$).

Proof. Let $\Phi \in dm(A)$. By Corollary 13, there exists $P \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ verifying $\langle e^P \mid \epsilon\rangle = 1, \langle e^P \mid x_0\rangle = \langle e^P \mid x_1\rangle = 0$ such that $\Phi = Z_{\sqcup\!\sqcup} e^P$. Using the factorization forms by Lyndon words, we get

$$\prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{\phi(l)\,\hat{l}} \quad = \quad \Big(\prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{\zeta(l)\,\hat{l}}\Big)\Big(\prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{p_l\,\hat{l}}\Big).$$

Expending the Hausdorff product and identifying the local coordinates, for any, $l \in \mathcal{L}ynX - \{x_0, x_1\}$, there exists $I_l \subset \{\lambda \in \mathcal{L}ynX - \{x_0, x_1\}$ s.t. $|\lambda| \le |l|\}$ and the coefficients $\{p'_u\}_{u \in I_l}$ belonging to A such that

$$\phi(l) = \sum_{u \in I_l} p'_u \, \zeta(u).$$

This belongs to $A[\{\zeta(l)\}_{l \in \mathcal{L}ynX} - \{x_0, x_1\}]$ and holds for any $P \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$. $\square$

With the notations of Definition 17 and by Corollary 14, we get in particular

Lemma 20. *For any $\Phi \in dm(A)$, by identifying the local coordinates (of second kind) on two members of the identities $\Psi = B(y_1)\Pi_Y\Phi$, or equivalently on $\Psi' = B'(y_1)\Pi_Y\Phi$, we get polynomial relations, of coefficients in A, among the convergent polyzêtas.*

Therefore,

Theorem 18. *While Φ describes $dm(A)$, the identities $\Psi = B(y_1)\Pi_Y\Phi$ describe the ideal of polynomial relations, of coefficients in A, among convergent polyzêtas.*

Moreover, if the Euler constant, γ, does not belong to A then these relations are algebraically independent on γ.

The computations on Section 4.3.2 is an example of such identities.

4.2.2 Concatenation of Chen generating series

As an example of the action of the differential Galois group of polylogarithms on their asymptotic expansions, we are interrested on the action of their monodromy group which is contained in $\mathrm{Gal}(DE)$.

The monodromies at 0 and 1 of Ł are given respectively by [35, 32]

$$\mathcal{M}_0\text{Ł} = \text{Ł}e^{2\mathrm{i}\pi\mathfrak{m}_0} \quad \text{and} \quad \mathcal{M}_1\text{Ł} = \text{Ł}(t)Z_{\sqcup\!\sqcup}^{-1}e^{-2\mathrm{i}\pi x_1}Z_{\sqcup\!\sqcup} = \text{Ł}e^{2\mathrm{i}\pi\mathfrak{m}_1}, \tag{100}$$

$$\text{where} \quad \mathfrak{m}_0 = x_0 \quad \text{and} \quad \mathfrak{m}_1 = \prod_{l \in \mathcal{L}ynX, l \neq x_0, x_1}^{\searrow} e^{-\zeta(\check{S}_l)\,\mathrm{ad}_{S_l}}(-x_1). \tag{101}$$

- If $C = 2\mathrm{i}\pi\mathfrak{m}_0$ then

$$\Phi = Z_{\sqcup\!\sqcup}e^{2\mathrm{i}\pi x_0}, \tag{102}$$

$$\Psi = \exp\biggl(\gamma y_1 - \sum_{k \ge 2}\zeta(k)\frac{(-y_1)^k}{k}\biggr)\Pi_Y Z_{\sqcup\!\sqcup} \tag{103}$$

$$= Z_{\text{⧢}}. \tag{104}$$

The monodromy at 0 consists in the multiplication on the right of $Z_{\sqcup\!\sqcup}$ by $e^{2\mathrm{i}\pi x_0}$ and does not modify $Z_{\text{⧢}}$.

- If $C = 2\mathrm{i}\pi\mathfrak{m}_1$ then

$$\begin{aligned} \Phi &= e^{-2\mathrm{i}\pi x_1} Z_{⧢}, && (105)\\ \Psi &= \exp\Big(\underbrace{(\gamma - 2\mathrm{i}\pi)}_{T:=} y_1 - \sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\Pi_Y Z_{⧢} && (106)\\ &= e^{-2\mathrm{i}\pi y_1} Z_{\text{⧢}\!\!\!+}. && (107)\end{aligned}$$

The monodromy at 1 consists in the multiplication on left of $Z_{⧢}$ and of $Z_{\text{⧢}\!\!\!+}$ by $e^{-2\mathrm{i}\pi x_1}$ and $e^{-2\mathrm{i}\pi y_1}$ respectively.

Remark 2. 1. The monodromies around the singularities of Ł could not allow, in this case, neither to introduce the factor $e^{\gamma x_1}$ on the left of $Z_{⧢}$ nor to eliminate the left factor $e^{\gamma y_1}$ in Z_γ (by putting $T = 0$, for example[21]).

2. By Proposition 5, we already saw that $Z_{⧢}$ is the concatenation of Chen generating series [10] $e^{x_0 \log \varepsilon}$ and then $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1 \log\varepsilon}$:

$$Z_{⧢} \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{x_1\log\varepsilon}\, S_{\varepsilon\rightsquigarrow 1-\varepsilon}\, e^{x_0\log\varepsilon}. \tag{108}$$

From (102) and (105), the action of the monodromy group gives

$$e^{x_1\, 2k_1\mathrm{i}\pi} Z_{⧢} e^{x_0\, 2k_0\mathrm{i}\pi} \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{x_1(\log\varepsilon + 2k_1\mathrm{i}\pi)}\, S_{\varepsilon\rightsquigarrow 1-\varepsilon}\, e^{x_0(\log\varepsilon+2k_0\mathrm{i}\pi)} \tag{109}$$

as being the concatenation of the Chen generating series $e^{x_0(\log\varepsilon+2k_0\mathrm{i}\pi)}$ (along circular path turning k_0 times around 0), then the Chen generating series $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, the Chen generating series $e^{x_1(\log\varepsilon+2k_1\mathrm{i}\pi)}$ (along circular path turning k_1 times around 1).

3. More generally, by Corollary 13, the action of the Galois differential group of polylogarithms states, for any Lie series C, the associator $\Phi = Z_{⧢} e^C$ is the concatenation of some Chen generating series e^C and $e^{x_0\log\varepsilon}$ and then the Chen generating series $S_{\varepsilon\rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1\log\varepsilon}$:

$$\Phi \underset{\varepsilon\to 0^+}{\widetilde{\quad}} e^{x_1\log\varepsilon}\, S_{\varepsilon\rightsquigarrow 1-\varepsilon}\, e^{x_0\log\varepsilon}\, e^C. \tag{110}$$

By construction (see Theorem 17) the associator Φ is then the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $S_{z_0\rightsquigarrow 1-z_0} e^C$, for $z_0 = \varepsilon \to 0^+$. Hence, by Corollary 7, we get the following

Corollary 15. *Let $\Phi \in dm(A)$. For any differential produced formal power series S over X (rational power series or polynomial), there exists a differential representation $(\mathcal{A}, f)$ such that :*

$$\begin{aligned}\langle \Phi \,\|\, S\rangle &= \sum_{w\in X^*} \langle \Phi \mid w\rangle\, \mathcal{A}(w)\circ f_{|_0}\\ &= \prod_{\ell\in\mathcal{L}ynX,\, l\neq x_0,x_1}^{\searrow} e^{\langle\Phi|\ell\rangle\, \mathcal{A}(\hat\ell)}\circ f_{|_0}.\end{aligned}$$

[21] Why ?

4.3 Algebraic combinatorial studies of the kernel of polyzêtas via the group of associators

4.3.1 Preliminary study

Let

$$A_1 = A\epsilon \oplus x_0 A\langle X\rangle x_1 \quad \text{and} \quad A_2 = A\epsilon \oplus (Y - \{y_1\})A\langle Y\rangle. \tag{111}$$

For $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y \Phi$. Let us introduce two algebra morphisms

$$\begin{array}{rcl} \phi : (A_1, ш) & \longrightarrow & A, \\ u & \longmapsto & \langle \Phi \mid u\rangle, \end{array} \qquad \begin{array}{rcl} \psi : (= A_2, ш_-) & \longrightarrow & A, \\ v & \longmapsto & \langle \Psi \mid v\rangle, \end{array} \tag{112}$$

and they verify respectively

$$\phi(\epsilon) = 1 \quad \text{and} \quad \phi(x_0) = \phi(x_1) = 0, \tag{113}$$

$$\psi(\epsilon) = 1 \quad \text{and} \quad \psi(y_1) = 0. \tag{114}$$

Lemma 21. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y \Phi$. Then*

$$\begin{array}{rrl} & \forall w \in Y^* - y_1 Y^*, & \psi(w) = \phi(\Pi_X w), \\ \textit{or equivalently}, & \forall w \in x_0 X^* x_1, & \phi(w) = \psi(\Pi_Y w). \end{array}$$

With the factorization of the monoids X^* (resp. Y^*), let $\{\hat{l}\}_{l \in \mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l \in \mathcal{L}ynY}$) be the dual basis of the Lyndon basis over X (resp. Y).

Lemma 22. *We have*

$$\Phi = \sum_{u \in X^*} \phi(u)\, u = \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\phi(l)\, \hat{l}} \quad \textit{and} \quad \Psi = \sum_{v \in Y^*} \psi(u)\, u = \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}}^{\searrow} e^{\psi(l)\, \hat{l}}.$$

With the notations in Lemma 22, we can get the following

Definition 20. *We put*

$$\mathcal{R} := \bigcap_{\Phi \in dm(A)} \ker \phi \quad (\textit{resp.} \quad \bigcap_{\substack{\Psi = B'(y_1)\Pi_Y \Phi \\ \Phi \in dm(A)}} \ker \psi).$$

Lemma 23. *Let $Q \in \mathbb{Q}[\mathcal{L}ynX]$ (resp. $\mathbb{Q}[\mathcal{L}ynY]$). For any $\Phi \in dm(A)$ and let $\Psi = B'(y_1)\Pi_Y \Phi$. Then*

$$\langle Q \parallel \Phi\rangle = 0 \iff Q \in \ker \phi \quad (\textit{resp.} \quad \langle Q \parallel \Psi\rangle = 0 \iff Q \in \ker \psi).$$

Or equivalently (see Definition 7),

$$Q \in \mathcal{R} \quad \iff \quad Q \textit{ is indiscernable over } dm(A).$$

Let Φ_1 and $\Phi_2 \in dm(A)$. By Corollary 13, for $i = 1$ or 2, there exists an unique $P_i \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ such that e^{-P_i} is well defined and $\Phi_i = Z_{\sqcup\!\sqcup} e^{P_i}$, or equivalently, $Z_{\sqcup\!\sqcup} = \Phi_1 e^{-P_1} = \Phi_2 e^{-P_2}$. Then, we get $\Phi_1 = \Phi_2 e^{P_1-P_2}$ and $\Phi_2 = \Phi_1 e^{P_2-P_1}$. By Lemma 19, it follows

Lemma 24. *Let Φ_1 and $\Phi_2 \in dm(A)$. For any convergent Lyndon word, l, there exists a finite set $I_l \subset \{\lambda \in \mathcal{L}ynX - \{x_0, x_1\}$ s.t. $|\lambda| \leq |l|\}$ and the coefficients $\{p'_{i,u}\}_{u\in I_l}$ and $\{p''_{i,u}\}_{u\in I_l}$, for $i = 1$ or 2, belonging to A such that*

$$\phi_i(l) = \sum_{u\in I_l} p'_{i,u}\,\zeta(u), \quad \textit{or equivalently}, \quad \zeta(l) = \sum_{u\in I_l} p''_{i,u}\,\phi_i(u).$$

There also exists the coefficients $\{p'_u\}_{u\in I_l}$ and $\{p''_u\}_{u\in I_l}$ belonging to A such that

$$\phi_1(l) = \sum_{u\in I_l} p'_u\,\phi_2(u), \quad \textit{or equivalently}, \quad \phi_2(l) = \sum_{u\in I_l} p''_u\,\phi_1(u).$$

Therefore, the $\{\phi_i(l)\}_{l\in\mathcal{L}ynX-\{x_0,x_1\}}$ (resp. $\{\psi_i(l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$), for $i = 1$ or 2, are also generators of the A-algebra generated by convergent polyzêtas.

4.3.2 Description of polynomial relations among coefficients of associator and irreducible polyzêtas

Since the identities of Corollary 14 (see also Corollary 12) hold for any pair of bases, in duality, compatible with factorization of the monoid X^* (resp. Y^*) then, by Corollary 14, one gets

Theorem 19. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y\Phi$. We have*

$$\prod_{l\in\mathcal{L}ynY, l\neq y_1}^{\searrow} e^{\psi(l)\,\hat{l}} = e^{\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}}\,\Pi_Y \prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{\phi(l)\,\hat{l}}.$$

If $\Phi = Z_{\sqcup\!\sqcup}$ and $\Psi = Z_{\uplus}$ then, for $\ell \in \mathcal{L}ynX - \{x_0, x_1\}$ (resp. $\mathcal{L}ynY - \{y_1\}$), one has $\zeta(l) = \phi(l)$ (resp. $\psi(l)$). Hence, one ontains (see also Corollary 12)

Theorem 20 (Bis repetita)**.**

$$\prod_{l\in\mathcal{L}ynY, l\neq y_1}^{\searrow} e^{\zeta(l)\,\hat{l}} = e^{\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}}\,\Pi_Y \prod_{l\in\mathcal{L}ynX, l\neq x_0,x_1}^{\searrow} e^{\zeta(l)\,\hat{l}}.$$

Corollary 16. *For $\ell \in \mathcal{L}ynY - \{y_1\}$ (resp. $\mathcal{L}ynX - \{x_0, x_1\}$), let $P_\ell \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$ (resp. $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$) be the decomposition of the polynomial $\Pi_X\hat{\ell} \in \mathbb{Q}\langle X\rangle$ (resp. $\Pi_Y\hat{\ell} \in \mathbb{Q}\langle Y\rangle$) in the Lyndon-PBW basis $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l\in\mathcal{L}ynY}$) and let $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}]$ (resp. $\mathbb{Q}[\mathcal{L}ynY - \{y_1\}]$) be its dual.*

Then one obtains

$$\Pi_X\ell - \check{P}_\ell \in \ker\phi \quad (\textit{resp.}\quad \Pi_Y\ell - \check{P}_\ell \in \ker\psi).$$

In particular, for $\phi = \zeta$ (resp. $\psi = \zeta$) then one also obtains

$$\Pi_X\ell - \check{P}_\ell \in \ker\zeta \quad (\textit{resp.}\quad \Pi_Y\ell - \check{P}_\ell \in \ker\zeta).$$

Moreover, for any $\ell \in \mathcal{L}ynY - \{y_1\}$ (resp. $\mathcal{L}ynX - \{x_0, x_1\}$), the polynomial $\Pi_Y\ell - \check{P}_\ell \in \mathbb{Q}\langle Y\rangle$ (resp. $\mathbb{Q}\langle X\rangle$) is homogenous of degree equal $\mid \ell \mid > 1$.

Proof. Since

$$\ell \in \mathcal{L}ynY \quad\Longleftrightarrow\quad \Pi_X\ell \in \mathcal{L}ynX - \{x_0\}$$

then identifying the local coordinates (of second kind) on the two members of each identity in Theorem 19, one obtains

$$\begin{array}{rl} \forall \ell \in \mathcal{L}ynY - \{y_1\} \subset Y^* - y_1Y^*, & \psi(\ell) = \phi(\check{P}_\ell), \\ (\text{resp. } \forall \ell \in \mathcal{L}ynX - \{x_0, x_1\} \subset x_0X^*x_1, & \phi(\ell) = \psi(\check{P}_\ell)). \end{array}$$

By Lemma 21, we get the expected result. □

With the notations of Corollary 16, we get the following

Definition 21. *Let Q_ℓ be the decomposition of the proper polynomial $\Pi_Y\ell - \check{P}_\ell$ (resp. $\Pi_X\ell - \check{P}_\ell$) in $\mathcal{L}ynY$ (resp. $\mathcal{L}ynX$). Let*

$$\begin{aligned} \mathcal{R}_Y &:= \{Q_\ell\}_{\ell \in \mathcal{L}ynY - \{y_1\}}, \\ \mathcal{R}_X &:= \{Q_\ell\}_{\ell \in \mathcal{L}ynX - \{x_0, x_1\}}. \end{aligned}$$

Definition 22. *Let Q_ℓ be the decomposition of the proper polynomial $\Pi_Y\ell - \check{P}_\ell$ (resp. $\Pi_X\ell - \check{P}_\ell$) in $\mathcal{L}ynY$ (resp. $\mathcal{L}ynX$). Let*

$$\begin{aligned} \mathcal{L}_{irr}Y &:= \{\ell \in \mathcal{L}ynY - \{y_1\} \mid Q_\ell = 0\}, \\ \mathcal{L}_{irr}X &:= \{\ell \in \mathcal{L}ynX - \{x_0, x_1\} \mid Q_\ell = 0\}. \end{aligned}$$

It follows that

Lemma 25. *We have*

$$\begin{aligned} (\mathbb{Q}[\mathcal{L}ynY - \{y_1\}], \,\text{⧢}\!\!\!\!\text{-}\,) &= (\mathcal{R}_Y, \,\text{⧢}\!\!\!\!\text{-}\,) \oplus (\mathbb{Q}[\mathcal{L}_{irr}Y], \,\text{⧢}\!\!\!\!\text{-}\,), \\ (\mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}], \sqcup\!\sqcup) &= (\mathcal{R}_X, \sqcup\!\sqcup) \oplus (\mathbb{Q}[\mathcal{L}_{irr}X], \sqcup\!\sqcup). \end{aligned}$$

Then we can state the following

Definition 23. *Any word w is said to be* irreducible *if and only if w belongs to $\mathcal{L}_{irr}Y$ (resp. $\mathcal{L}_{irr}X$).*

For any $Q \in \mathbb{Q}[\mathcal{L}_{irr}X]$, by Corollary 7, there exists a differential representation $(\mathcal{A}, f)$ such that Q can be *finitely* factorized (see also Corollary 15) :

$$Q = \sum_{w \in X_{irr}^*} \mathcal{A}(w) \circ f\, w = \prod_{\ell \in \mathcal{L}_{irr}X, \text{finite}}^{\searrow} e^{\mathcal{A}(\hat{\ell})\, \ell} \circ f, \tag{115}$$

where X_{irr}^* denotes the set of words obtaining by shuffling on $\mathcal{L}_{irr}X$.

Lemma 26. *Any proper polynomial* $P \in (\mathbb{Q}[\mathcal{L}_{irr}X], ⧢)$ *(resp.* $(\mathbb{Q}[\mathcal{L}_{irr}Y], \boxplus)$*) is indiscernable over Chen generating series* $\{e^{t\,x}\}_{x \in X}^{t \in \mathbb{R}}$ *:*

$$\langle P \parallel e^{t\,x_0} \rangle = \langle P \parallel e^{t\,x_1} \rangle = 0 \quad (\textit{resp.} \quad \langle P \parallel e^{t y_1} \rangle = 0).$$

Proof. By construction, x_0 and $x_1 \notin \mathcal{L}_{irr}X$ (resp. $y_1 \notin \mathcal{L}_{irr}X$). For any $n > 1, x_0^n$ and x_1^n (resp. y_1^n) are not Lyndon words then they do not belong to $\mathcal{L}_{irr}X$ (resp. $\mathcal{L}_{irr}X$). Therefore, for any $n \geq 0$, one has

$$\langle P \mid x_0^n \rangle = \langle P \mid x_1^n \rangle = 0 \quad (\text{resp.} \quad \langle P \mid y_1^n \rangle = 0).$$

Using the expansion of the exponential, we find the expected result. □

Lemma 27. *Let* $\Phi \in dm(A)$ *and let* $t, \in \mathbb{R}, x \in X$*. For any proper polynomial* $P \in (\mathbb{Q}[\mathcal{L}_{irr}X], ⧢)$*, if* $\langle P \parallel \Phi \rangle = 0$ *then we have*

$$\langle P \parallel \Phi e^{t\,x} \rangle = 0 \quad \textit{and} \quad \langle P \parallel e^{t\,x} \Phi \rangle = 0.$$

Proof. Since $P \in (\mathbb{Q}[\mathcal{L}_{irr}X], ⧢)$ and P is proper then, by Lemma 26, for any $t \in \mathbb{R}$ and for any $x \in X$, we have $\langle P \parallel e^{t\,x} \rangle = 0$ and then $\langle P \parallel \Phi e^{t\,x} \rangle = 0$.

Since $\mathrm{supp}(P) \subset x_0 X^* x_1$, we also have $\langle P \parallel e^{t\,x_0} \Phi \rangle = \langle P \triangleright e^{t\,x_0} \parallel \Phi \rangle = 0$. Next, for any $\Phi \in dm(A)$, there exists e^C such that $e^{t\,x_1}\Phi = e^{t\,x_1} Z_{⧢} e^C$ and, by Proposition 5, we get

$$e^{t\,x_1}\Phi \quad \widetilde{_{\varepsilon \to 0^+}} \quad e^{x_1(t + \log \varepsilon)}\, S_{\varepsilon \rightsquigarrow 1-\varepsilon}\, e^{x_0 \log \varepsilon}\, e^C.$$

Hence, there exists a Chen generating series $C_{z \rightsquigarrow 1-z_0}$ and $S_{z_0 \rightsquigarrow 1-z_0}$ such that we get the following asymptotic behaviour (see Section 4.2.2)

$$e^{t\,x_1}\Phi \quad \widetilde{_{\varepsilon \to 0^+}} \quad C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z}\, e^C$$

and the following concatenation holds [10] (see Formula (64))

$$\begin{array}{rcl} C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} &=& S_{z_0 \rightsquigarrow 1-z_0}, \\ \iff \qquad C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C &=& S_{z_0 \rightsquigarrow 1-z_0} e^C. \end{array}$$

Since $P \in \mathbb{Q}[\mathcal{L}_{irr}X]$ then, by Corrolary 7, there exists a differential representation $(\mathcal{A}, f)$ such that $P = \sigma f_{|_0}$. Applying $\langle \sigma f_{|_0} \parallel \bullet \rangle$ to the two sides of the previous equality, one has (see Theorem 9)

$$\langle \sigma f_{|_0} \parallel C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C \rangle \quad = \quad \langle \sigma f_{|_0} \parallel S_{z_0 \rightsquigarrow 1-z_0} e^C \rangle.$$

Thus, for $z_0 = \varepsilon \to 0^+$, one obtains (see Corollary 3)

$$\langle \sigma f_{|_0} \| e^{t\, x_1} \Phi \rangle \quad \widetilde{_{\varepsilon \to 0^+}} \quad \langle \sigma f_{|_0} \| \Phi \rangle.$$

By assumption, $\langle \sigma f_{|_0} \| \Phi \rangle = \langle P \| \Phi \rangle = 0$ then we get the expected result. $\square$

For any $P \in (\mathbb{Q}[\mathcal{L}_{irr} X], \sqcup\!\sqcup)$ (resp. $(\mathbb{Q}[\mathcal{L}_{irr} Y], \uplus)$), $\zeta(P)$ is a polynomial on irreducible polyzêtas. One also has

Lemma 28. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y \Phi$. We have $\mathcal{R}_Y \subseteq \ker \psi$ and $\mathcal{R}_X \subseteq \ker \phi$. In particular, $\mathcal{R}_Y \subseteq \ker \zeta$ and $\mathcal{R}_X \subseteq \ker \zeta$.*

Proposition 19. *We have $\mathcal{R}_X \subseteq \mathcal{R}$ (resp. $\mathcal{R}_Y \subseteq \mathcal{R}$).*

Proposition 20. *For any proper polynomial $Q \in \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$),*

$$Q \in \mathcal{R} \quad \Longleftrightarrow \quad Q = 0.$$

Proof. If $Q = 0$ then since, for any $\Phi \in dm(A), \phi$ is an algebra homorphism then $\phi(Q) = 0$. Hence, $Q \in \ker \phi$ and then $Q \in \mathcal{R}$.

Conversely, if $Q \in \mathcal{R}$ then, for any $\Phi \in dm(A)$, we get $\langle Q \| \Phi \rangle = 0$. That means that Q is indiscernable over $dm(A)$. Let $\mathcal{H}$ defined as being the monoid generated by $dm(A)$ and by the Chen's generating series $\{e^{t\, x}\}_{x \in X}^{t \in \mathbb{R}}$.

By Lemma 9, Q is continuous over $\mathcal{H}$ and by Lemma 27, it is indiscernable over $\mathcal{H}$. By Proposition 7, the expected result follows. $\square$

Therefore, by the propositions 19 and 20, we obtain

Theorem 21. *We have $\mathcal{R} = \mathcal{R}_X$ (resp. $\mathcal{R}_Y$).*

Lemma 29. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y \Phi$. Let $Q \in \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$) such that $\langle \Phi \| Q \rangle = 0$ (resp. $\langle \Psi \| Q \rangle = 0$). Then $Q = 0$.*

Proof. Let $\mathcal{H}$ defined as being the monoid generated by Φ and by Chen generating series $\{e^{t\, x}\}_{x \in X}^{t \in \mathbb{R}}$. By assumption, $\langle \Phi \| Q \rangle = 0$ and by Lemma 27, Q is then indiscernable over $\mathcal{H}$. Finally, by Proposition 7, it follows that $Q = 0$. $\square$

Proposition 21. *For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y \Phi$. We get*

$$\ker \phi = \mathcal{R}_X \quad \textit{(resp.} \quad \ker \psi = \mathcal{R}_Y).$$

In particular, $\ker \zeta = \mathcal{R}_X$ (resp. $\ker \zeta = \mathcal{R}_Y$).

Proof. By Lemma 28, $\mathcal{R}_X$ and $\mathcal{R}_Y$ are included in $\ker \phi$ and $\ker \psi$ respectively.

Conversely, two cases can occur (see Lemma 25) :

1. Case $Q \notin \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$). By Lemma 25, $Q \equiv_{\mathcal{R}_X} Q_1$ (resp. $Q \equiv_{\mathcal{R}_Y} Q_1$) such that $Q_1 \in \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$) and $\phi(Q_1) = 0$ (resp. $\psi(Q_1) = 0$). This case is then reduced to the following

2. Case $Q \in \mathbb{Q}[\mathcal{L}_{irr} X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr} Y]$). Using Lemma 29, we have $Q \equiv_{\mathcal{R}_X} 0$ (resp. $Q \equiv_{\mathcal{R}_Y} 0$).

Then, $\mathcal{R}_X$ (resp. $\mathcal{R}_Y$) contains $\ker \phi$ (resp. $\ker \psi$) respectively. $\square$

Hence, for any $\Phi \in dm(A)$, by Proposition 21, one has

$$\ker \phi = \ker \zeta = \mathcal{R}_X. \tag{116}$$

That means, for any irreducible Lyndon words $l \neq l'$,

$$\phi(l) = \phi(l') \quad \Longleftrightarrow \quad \zeta(l) = \zeta(l'). \tag{117}$$

Proposition 22. *The $\mathbb{Q}$-algebra $\mathcal{Z}$ is freely generated, over $\mathbb{Q}$, by irreducible polyzêtas.*

Proof. By Radford's theorem [45], one just needs to prove the result for Lyndon words : for any $\ell \in \mathcal{L}ynY - \{y_1\}$, if $\Pi_X \ell = \check{P}_\ell$ then one gets the conclusion else one has $\Pi_X \ell - \check{P}_\ell \in \ker \zeta$. Hence, $\zeta(\ell) = \zeta(\check{P}_\ell)$.

Since $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}]$ then $\check{P}_\ell$ is a polynomial on Lyndon words of degree less or equal $\mid \ell \mid$. For each Lyndon word, over X, does appear in this decomposition of $\check{P}_\ell$, after applying Π_Y, one makes the same procedure until one gets Lyndon words in $\mathcal{L}_{irr}Y$.

The same treatment works for any $\ell' \in \mathcal{L}_{irr}X$. $\square$

Hence, let us state the following

Lemma 30. *Let $\Phi \in dm(A)$. Let us define the map $\varphi : \mathcal{Z} \longrightarrow A$ as follows*

$$\forall l \in \mathcal{L}_{irr}X, \quad \varphi(\zeta(l)) \quad := \quad \phi(l).$$

Then φ is an algebra homomorphism and $\{\varphi(\zeta(l))\}_{l \in \mathcal{L}_{irr}X}$ are generators of the $\mathbb{Q}$-algebra A.

Hence, for any $\theta \in \mathcal{Z}$, there are $\{\alpha_{l_1,\ldots,l_n}\}_{l_1,\ldots,l_n \in \mathcal{L}_{irr}X}^{n \in \mathbb{N}}$ in A such that

$$\varphi(\theta) \quad = \quad \sum_{n \geq 0} \sum_{l_1,\ldots,l_n \in \mathcal{L}_{irr}X} \alpha_{l_1,\ldots,l_n} \, \varphi(\zeta(l_1)) \ldots \varphi(\zeta(l_n)). \tag{118}$$

In particular, since for any $w \in X^*$, $\zeta_{\sqcup\!\sqcup}(w)$ belongs to $\mathcal{Z}$ (see Corollary 9) then $\varphi(\zeta_{\sqcup\!\sqcup}(w))$ is well defined and, with the notations of Lemma 30, $\varphi(\zeta_{\sqcup\!\sqcup}(w))$ can be expressed as polynomial on convergent polyzêtas with coefficients in A :

Lemma 31. *For any $w \in X^*$, one has*

$$\varphi(\zeta_{\sqcup\!\sqcup}(w)) \quad = \quad \sum_{u,v \in X^*, uv = w} \langle e^C \mid v \rangle \, \zeta_{\sqcup\!\sqcup}(u).$$

Proof. Identifying the coefficients on $\Phi = Z_{\sqcup\!\sqcup} e^C$, we get the expected result. $\square$

Finally, we can state the following

Theorem 22. *For any $\Phi \in dm(A)$, there exists an unique algebra homomorphism $\varphi : \mathcal{Z} \longrightarrow A$ such that Φ is computed from $Z_{\sqcup\!\sqcup}$ by applying φ to each coefficient :*

$$\Phi = \sum_{w \in X^*} \varphi(\zeta_{\sqcup\!\sqcup}(w)) \, w = \prod_{l \in \mathcal{L}ynX, l \neq x_0, x_1}^{\searrow} e^{\varphi(\zeta(l)) \, \hat{l}}.$$

Remark 3. 1. In this work, neither the question to decide any real number belongs to $\mathcal{Z}$ or not nor the question to explicit effectively the coefficients $\{\alpha_{l_1,\ldots,l_n}\}_{l_1,\ldots,l_n\in\mathcal{L}_{irr}X}^{n\in\mathbb{N}}$ in (118), are considered.

2. Now, by considering the commutative indeterminates $t_1, t_2, t_3, \ldots$, let A be the $\mathbb{Q}$-algebra obtained by specializing $\mathbb{Q}[t_1, t_2, t_3, \ldots]$ at $t_1 = \mathrm{i}\pi$:

$$A = \mathbb{Q}[\mathrm{i}\pi, t_2, t_3, \ldots]. \tag{119}$$

Neither the Lie exponential series $Z_{\sqcup\!\sqcup}e^{2\mathrm{i}\pi x_0}$ and $Z_{\sqcup\!\sqcup}e^{2\mathrm{i}\pi x_1}$, in Section 4.2.2, nor $Z_{\sqcup\!\sqcup}e^{\mathrm{i}\pi x_0}$ and $Z_{\sqcup\!\sqcup}e^{\mathrm{i}\pi x_1}$, in Proposition 6, do belong to $dm(A)$ but they belong to $\mathrm{Gal}(DE)$. Using Baker–Campbell–Hausdorff formula [6] in Proposition 6 we get, at orders 2 and 3 as examples, the famous Euler's formulaes saying $\zeta(2)$ is algebraic over A :

$$\zeta(2) + \frac{(\mathrm{i}\pi)^2}{6} \quad = \quad 0 \quad \text{(order 2)}, \tag{120}$$

$$\zeta(3) - \zeta(2,1) \quad = \quad 0 \quad \text{(order 3, imaginary part)}. \tag{121}$$

Therefore, the first comming in mind homomorphism $\varphi : \mathcal{Z} \longrightarrow A$ maps, at least $\zeta(2)$ to $\varphi(\zeta(2)) = \pi^2/6$.

3. For this reason, in [30], we have to consider the $\mathbb{Q}$-algebra generated by $\mathrm{i}\pi$ and by other irreducible polyzêtas obtained in [32, 34, 3, 49] (and such algebra is denoted in this work by A). This algebra came up from the studies of monodromies [32, 35], as already shown in (100), and the Kummer type functional equations of polylogarithms [32, 36], as already shown in (54)–(56). In particular, by (56), we get for example [36, 32],

$$\begin{aligned}\mathrm{Li}_{2,1}\frac{1}{t} \quad = \quad & -\frac{(\mathrm{i}\pi)^2}{2}\log t + \mathrm{i}\pi(\zeta(2) - \frac{\log^2 t}{2} - \mathrm{Li}_2\, t) \\ & - \mathrm{Li}_{2,1}\, t + \mathrm{Li}_3\, t - \log t\,\mathrm{Li}_2\, t + \zeta(3) - \frac{\log^3 t}{6}.\end{aligned} \tag{122}$$

Specialization at $t = 1$, the real part of this leads again to the Euler's indentity (121).

5 Concluding remarks : a complete description of $\ker\zeta$ and the structure of polyzêtas

Once again, for (see Definition 18, Lemma 25 and the definitions 21, 22)

$$\begin{aligned}
(C_1, \sqcup\!\sqcup) &= (\mathbb{Q}\epsilon \oplus x_0\mathbb{Q}\langle X\rangle x_1, \sqcup\!\sqcup) && (123)\\
&\cong \mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}], \sqcup\!\sqcup) && (124)\\
&= (\mathcal{R}_X, \sqcup\!\sqcup) \oplus (\mathbb{Q}[\mathcal{L}_{irr}X], \sqcup\!\sqcup), && (125)\\
(C_2, \uplus) &= (\mathbb{Q}\epsilon \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle, \uplus) && (126)\\
&\cong (\mathbb{Q}[\mathcal{L}ynY - \{y_1\}], \uplus) && (127)\\
&= (\mathcal{R}_Y, \uplus) \oplus (\mathbb{Q}[\mathcal{L}_{irr}Y], \uplus), && (128)
\end{aligned}$$

we have [32, 33]

$$(C_1, ш) \quad \cong \quad (C_2, ⧢). \tag{129}$$

5.1 Structure of polyzêtas

Let us consider again the following algebra morphism (see Proposition 15)

$$\zeta \quad : \quad \begin{matrix} (C_2, ⧢) \\ (C_1, ш) \end{matrix} \quad \longrightarrow \quad (\mathbb{R}, .) \tag{130}$$

$$\begin{matrix} y_{r_1}\ldots y_{r_k} \\ x_0x_1^{r_1-1}\ldots x_0x_1^{r_k-1} \end{matrix} \quad \longmapsto \quad \sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{r_1}\ldots n_k^{r_k}}, \tag{131}$$

and let us recall $\mathcal{Z}$ (resp. $\mathcal{Z}'$) is the $\mathbb{Q}$-algebra (resp. $\mathbb{Q}[\gamma]$-algebra) generated by $\{\zeta(l)\}_{l\in\mathcal{L}ynX-\{x_0,x_1\}}$ (resp. $\{\zeta(l)\}_{l\in\mathcal{L}ynY}-\{y_1\}$, see Definisition 3).

Lemma 32. *The image of the algebra morphism ζ is $\mathcal{Z}$.*

We have, as consequences of the propositions 22 and 21,

$$\mathrm{Im}\,\zeta = \zeta(\mathbb{Q}[\mathcal{L}_{irr}Y]) \quad \text{and} \quad \ker\zeta = \mathcal{R}_Y, \tag{132}$$

$$(\text{resp.} \quad \mathrm{Im}\,\zeta = \zeta(\mathbb{Q}[\mathcal{L}_{irr}X]) \quad \text{and} \quad \ker\zeta = \mathcal{R}_X). \tag{133}$$

Now, let us make precise the structure of $\mathcal{Z}$.

Theorem 23 (Structure of polyzêtas)**.** *The $\mathbb{Q}$-algebra generated by convergent polyzêtas, $\mathcal{Z}$, is isomorphic to the graded algebra $(C_1/\mathcal{R}_X, ш)$, or equivalently, $(C_2/\mathcal{R}_Y, ⧢)$.*

Proof. Since $\ker\zeta$ is an ideal generated by the homogenous polynomials (see Corollary 16 and Formula (132)) then $C_1/\mathcal{R}_X$ and $C_2/\mathcal{R}_Y$ are graded [6]. ☐

5.2 A conjecture by Pierre Cartier

Definition 24 ([8, 43])**.** *Let $DM(A)$ be the set of $\Phi \in A\langle\langle X\rangle\rangle$ such that*

$$\langle\Phi \mid \epsilon\rangle = 1, \quad \langle\Phi \mid x_0\rangle = \langle\Phi \mid x_1\rangle = 0, \quad \Delta_ш\Phi = \Phi\otimes\Phi$$

and such that, for

$$\bar{\Psi} = \exp\Big(-\sum_{n\geq 2}\langle\Pi_Y\Phi \mid y_n\rangle\frac{(-y_1)^n}{n}\Big)\Pi_Y\Phi \in A\langle\langle Y\rangle\rangle,$$

then $\Delta_{⧢}\bar{\Psi} = \bar{\Psi}\otimes\bar{\Psi}$.

Since $DM(A)$ contains already $Z_ш$ then for any $\Phi \in DM(A)$, Φ is group-like, for the coproduct $\Delta_ш$. By the group of associators theorem (Theorem 17),

there exists $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ satisfying $\langle e^C \mid \epsilon\rangle = 1$, $\langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0$ such that $\Phi = Z_{\sqcup\!\sqcup} e^C$ and such that

$$\Psi = B'(y_1)\Pi_Y\Phi = \exp\Big(-\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\Pi_Y\Phi, \tag{134}$$

$$\bar{\Psi} = \exp\Big(-\sum_{n\geq 2}\langle\Pi_Y\Phi \mid y_n\rangle\frac{(-y_1)^n}{n}\Big)\Pi_Y\Phi. \tag{135}$$

Hence, Ψ is group-like and $\bar{\Psi}$ must be also group-like, for the coproduct $\Delta_{\uplus}$. If such a Lie series C exists then it is unique due to the fact that $e^C = \Phi Z_{\sqcup\!\sqcup}^{-1}$ and it is group-like, for the coproduct $\Delta_{\sqcup\!\sqcup}$.

Corollary 17 (conjectured by Cartier, [8])**.** *For any $\Phi \in DM(A)$, there exists an unique algebra homomorphism*[22] $\bar{\varphi} : \mathcal{Z} \longrightarrow A$ *such that Φ is computed from $Z_{\sqcup\!\sqcup}$ by applying $\bar{\varphi}$ to each coefficient.*

Proof. By Theorem 22, use the fact $DM(\mathcal{Z}) \subseteq DM(A) \subseteq dm(A)$. □

5.3 Arithmetical nature of γ

By Theorem 18, under the assumption that the Euler constant, γ, does not belong to a commutative $\mathbb{Q}$-algebra A then γ does not verify any polynomial with coefficients in A among the convergent polyzêtas. It follows then,

Corollary 18. *If $\gamma \notin A$ then it is transcendental over the A-algebra generated by the convergent polyzêtas.*

Or equivalently, by contraposition,

Corollary 19. *If there exists a polynomial relation with coefficients in A among the Euler constant, γ, and the convergent polyzêtas then $\gamma \in A$.*

Therefore,

Corollary 20. *If the Euler constant, γ, does not belong to A then γ is not algebraic over A.*

Using Corollary 20, with $A = \mathbb{Q}$, it follows that

Corollary 21. *The Euler constant, γ, is not an algebraic irrational number.*

Corollary 22. *The Euler constant, γ, is a rational number.*

Proof. Let us prove that in three steps[23] :

1. Since γ verifies the equation $t^2 - \gamma^2 = 0$ then γ is algebraic over $\mathbb{Q}(\gamma^2)$.

[22] See Remark 3 (point 2) to have an example of $\bar{\varphi}$.
[23] This part is obtained after prolonged discussions with Michel Waldschmidt.

2. If γ is transcendental over $\mathbb{Q}$ then $\gamma \notin \mathbb{Q}(\gamma^2)$. Using Corollary 20, with $A = \mathbb{Q}(\gamma^2)$, γ is not algebraic over $A = \mathbb{Q}(\gamma^2)$. It contradicts the previous assersion (*i.e.* step 1.). Hence, γ is not transcendental over $\mathbb{Q}$.

3. Thus, by Corollary 21, it remains that γ is rational over $\mathbb{Q}$.

$\square$

Remark 4. 1. In the same spirit of Theorem 15, let $\zeta^T_{\stuffle}$ be the regularization morphism[24] from $(\mathbb{Q}\langle\langle Y\rangle\rangle, \stuffle)$ to $(\mathbb{R}, .)$ mapping y_1 to T. Let $Z^T_{\stuffle}$ be the noncommutative generating series of polyzêtas regulariszed with respect to $\zeta^T_{\stuffle}$. Thus, as in Theorem 15 and by infinite factorization by Lyndon words, we also get

$$Z^T_{\stuffle} := \sum_{w \in X^*} \zeta^T_{\stuffle}(w)\, w = e^{Ty_1} Z_{\stuffle}. \tag{136}$$

2. Now let $B^T(y_1) = e^{Ty_1} B'(y_1)$ be the regularization, for $N \to +\infty$ and with respect to $\zeta^T_{\stuffle}$, of the power series $\mathrm{Const}(N)$ given in (60). As in Corollary 12, we always get

$$Z^T_{\stuffle} = B^T(y_1) \Pi_Y Z_{\shuffle} \quad \Longleftrightarrow \quad Z_{\stuffle} = B'(y_1) \Pi_Y Z_{\shuffle}. \tag{137}$$

   Hence, roughtly speaking, for the quasi-shuffle product, the symbolic regularization to T is also "equivalent" to the regularization to 0.

3. Again, as in Corollary 18, if $T \notin A$ then $T \notin \bar{A}$.

   In contrario, as in Corollary 19, if there exists a polynomial relation with coefficients in A among T and convergent polyzêtas then $T \in A$.

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